Risk & Asset Allocation (Spring)
Exercise for Week 1

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Let us consider the expected shortfall index of satisfaction for a very simple portfolio: $\alpha$ shares in an asset whose value today is $p > 0$ and whose horizon value $P$ is lognormal.

Let us assume that the objective measure is profit; therefore in Meucci’s notation, we have

$$
\Psi_\alpha = \alpha M
= \alpha (P - p)
= \alpha (g(X) - p)
= \alpha p (e^X - 1)
$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean $\mu$ and variance $\Sigma > 0$. The index of satisfaction is

$$
S(\alpha) = \frac{1}{1 - c} \int_0^{1 - c} Q_{\Psi_\alpha}(q) \, dq
$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.
We proceed to evaluate the exact version by considering the CDF of the objective.

\[ F_{\Psi_\alpha}(z) = P\{\Psi_\alpha < z\} \]
\[ = P\{\alpha p (e^X - 1) < z\} \]
\[ = P\left\{ X \text{ sgn } \alpha < \log \left( 1 + \frac{z}{\alpha p} \right) \text{ sgn } \alpha \right\} \]
\[ = P\left\{ \frac{X - \mu}{\sqrt{\Sigma}} \text{ sgn } \alpha < \frac{\log \left( 1 + \frac{z}{\alpha p} \right) - \mu}{\sqrt{\Sigma}} \text{ sgn } \alpha \right\} \]
\[ = \Phi \left( \frac{\log \left( 1 + \frac{z}{\alpha p} \right) - \mu}{\sqrt{\Sigma} \text{ sgn } \alpha} \right) \]

where \( \Phi(\cdot) \) is the CDF of a standard normal.

The quantile, which is the inverse of the CDF, is therefore

\[ Q_{\Psi_\alpha}(q) = \alpha p \left( e^\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q) - 1 \right) \]

So can proceed to evaluate the index of satisfaction.

\[ S(\alpha) = \frac{1}{1 - c} \int_0^{1-c} \alpha p \left( e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) dq \]
\[ = \alpha p \left( \frac{1}{1 - c} \int_0^{1-c} e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \Phi^{-1}(q)} dq - 1 \right) \]
\[ = \alpha p \left( \frac{1}{1 - c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \phi(z)} dz - 1 \right) \]

where the last line is achieved by the change of variable \( z = \Phi^{-1}(q) \) and \( \phi(z) = \Phi'(z) \) is the density of a standard normal.

Since

\[ e^{\mu + \text{sgn } \alpha \sqrt{\Sigma} \phi(z)} = e^{\mu + \frac{1}{2} \Sigma \phi \left( z - \text{ sgn } \alpha \sqrt{\Sigma} \right)} \]

we have the final result,

\[ S(\alpha) = \alpha p \left( e^{\mu + \frac{1}{2} \Sigma} \frac{1}{1 - c} \Phi \left( \Phi^{-1}(1-c) - \text{ sgn } \alpha \sqrt{\Sigma} \right) - 1 \right) \quad (1) \]

## 2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

\[ S(\alpha) \approx \alpha p \left( \mu - \text{ sgn } \alpha \frac{\phi \left( \Phi^{-1}(1-c) \right)}{1 - c} \sqrt{\Sigma} \right) \quad (2) \]
Let us spend a moment interpreting this. An investor will be more satisfied to be long \((\alpha > 0)\) if the asset has a positive expected return \((\mu > 0)\), and short \((\alpha < 0)\) if the asset has a negative expected return \((\mu < 0)\). In contrast, positive variance diminishes satisfaction for any non-zero position.

This all seems quite reasonable for a rational index of satisfaction.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the objective \(\Psi_{\alpha}\). In a Delta-Gamma setting, we can replace the objective by the quadratic

\[
\Psi_{\alpha} = \alpha p \left(e^X - 1\right) \approx \alpha p \left(X + \frac{1}{2}X^2\right)
\]

hence \(\Theta_{\alpha} = 0\), \(\Delta_{\alpha} = \alpha p\), and \(\Gamma_{\alpha} = \alpha p\). Let us define a new objective to represent this approximation.

\[
\Xi_{\alpha} = \alpha p \left(X + \frac{1}{2}X^2\right)
\]

Is is straight-forward to work out that the first several central moments of this are

\[
\mathbb{E}(\Xi_{\alpha}) = \alpha p \left(\mu + \frac{1}{2}\mu^2 + \frac{1}{2}\Sigma\right)
\]

\[
\text{Sd}(\Xi_{\alpha}) = |\alpha|p\sqrt{\Sigma}\sqrt{(1 + \mu)^2 + \frac{1}{2}\Sigma}
\]

\[
\text{Sk}(\Xi_{\alpha}) = 3 \text{sgn}\alpha \sqrt{\Sigma} \frac{(1 + \mu)^2 + \frac{1}{2}\Sigma}{((1 + \mu)^2 + \frac{1}{2}\Sigma)^{3/2}}
\]

The third-order Cornish-Fisher expansion for expected shortfall in general is

\[
S(\alpha) \approx \mathbb{E}(\Xi_{\alpha}) + \text{Sd}(\Xi_{\alpha}) \left(z_1 + \frac{z_2 - 1}{6} \text{Sk}(\Xi_{\alpha})\right)
\]
with coefficients
\[
zh_1 = \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q) \, dq = - \frac{\phi(\Phi^{-1}(1-c))}{1-c} \\
zh_2 = \frac{1}{1-c} \int_0^{1-c} \Phi^{-1}(q)^2 \, dq = 1 - \frac{\phi(\Phi^{-1}(1-c))}{1-c} \Phi^{-1}(1-c)
\]
depending on the confidence level \(c < 1\).

Putting this together, we get a third expression for the index of satisfaction.

\[
S(\alpha) \approx \alpha p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) - |\alpha| p \frac{\phi(\Phi^{-1}(1-c))}{1-c} \sqrt{\Sigma} \\
\cdot \left( \sqrt{(1+\mu)^2 + \frac{1}{2} \Sigma} + \frac{1}{2} \text{sgn } \alpha \frac{(1+\mu)^2 + \frac{1}{2} \Sigma \Phi^{-1}(1-c) \sqrt{\Sigma}}{(1+\mu)^2 + \frac{1}{2} \Sigma} \right)
\]

This result agrees with (2) to lowest order in \(\mu\) and \(\Sigma\).

\section{4 Modeling Default}

Our horizon asset value \(P\) is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

\[
\Psi'_{\alpha} = \alpha p \left( Ye^X - 1 \right)
\]

where \(X \sim \mathcal{N}(\mu, \Sigma)\) as before\(^2\), but now we add an independent default indicator \(Y \sim \text{Bern}(1-h)\) for default probability \(h\).

\(^1\)The trick to these integrals is to realize that \(\phi'(z) = -z\phi(z)\).
\(^2\)Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.