Risk & Asset Allocation
Case Solution for Spring 1

John A. Dodson
January 23, 2013

Say $X$ is the invariant random variable that will drive market prices $P_{T+\tau} \triangleq g(X; p_T)$ between today, time $T$, and the next decision date, time $T + \tau$ with an investment horizon of $\tau$ (all time measured in years) and say that we have used historical data to estimate the parameters of the characterization of the random variable $\tilde{X}$ under a different (usually shorter) horizon, $\tilde{\tau}$; e.g., from a timeseries sample $(p_T, p_{T-\tilde{\tau}}, p_{T-2\tilde{\tau}}, \ldots)$.

Furthermore, say our invariant mapping is additive in the sense that

$$g(\tilde{x}_1; g(\tilde{x}_2, p_{T-2\tilde{\tau}})) = g(\tilde{x}_1 + \tilde{x}_2, p_{T-2\tilde{\tau}})$$

which is true, for example, with the continuous version of total return (although the simple version of total return is often an adequate approximation).

Then projecting $X$ from $\tilde{X}$ essentially involves the transformation

$$\tilde{X} \overset{\text{i.i.d. copies}}{\|} \tilde{X} + \tilde{X} + \cdots$$

We know from the properties of characteristic functions that as long as increments are independent and identically distributed,

$$\phi_X(\omega) = \phi_{\tilde{X}}(\omega)^\tilde{\tau}$$

If the first two moments exist, we also know that

$$EX = -i \left. \frac{d\phi_X}{d\omega'} \right|_0$$

$$= -i \frac{\tilde{\tau}}{\tau} \left. \frac{\phi_{\tilde{X}}}{d\omega'} \right|_0$$

$$= \frac{\tau}{\tilde{\tau}} E\tilde{X}$$

and

$$E(XX') = - \left. \frac{d^2\phi_X}{d\omega'd\omega} \right|_0$$

$$= - \frac{\tilde{\tau}}{\tau} \left( \frac{\tau}{\tilde{\tau}} - 1 \right) (\phi_{\tilde{X}})^{\tilde{\tau}-2} \left. \frac{d\phi_{\tilde{X}}}{d\omega'} \right|_0 - \tau \left. \left( \phi_{\tilde{X}} \right)^{\tilde{\tau}-1} \frac{d^2\phi_{\tilde{X}}}{d\omega'd\omega} \right|_0$$

$$= \frac{\tau}{\tilde{\tau}} \left( \frac{\tau}{\tilde{\tau}} - 1 \right) E\tilde{X} E\tilde{X}' + \frac{\tau}{\tilde{\tau}} E\left(\tilde{X}\tilde{X}'\right)$$

$$= EXEX' + \frac{\tau}{\tilde{\tau}} \left( E\left(\tilde{X}\tilde{X}'\right) - E\tilde{X} E\tilde{X}' \right)$$
Since the covariance is defined as
\[
\text{cov} \tilde{X} = \mathbb{E} (XX') - \mathbb{E} X \mathbb{E} X'
\]
we have that
\[
\text{cov} X = \tau \text{cov} \tilde{X}
\]
Furthermore, since
\[
\text{std} \tilde{X} = \text{diag} \sqrt{\text{diag} \text{diag} \text{cov} \tilde{X}}
\]
we have the “square-root rule” for time-scaling market invariants.
\[
\text{std} X = \sqrt{\tau} \text{std} \tilde{X}
\]
This is valid regardless of the distribution of \( \tilde{X} \) (as long as it has two moments).
Note that in general \( X \) will not belong to the same family of random variables as \( \tilde{X} \).