1 Introduction

Nonlinear asymmetric GARCH (generalized autoregressive conditional heteroskedasticity), is a model introduced by Robert Engle and Victor Ng in 1993, building upon the models for timeseries conditional heteroskedasticity first put forward in the 1980’s by Robert Engle for which he was ultimately awarded the 2003 Nobel Prize in economics.

2 Specification

Following the notation in [1], say we have periodic timeseries observations of some asset’s total return, \{y_1, y_2, \ldots, y_T\} and a model for the conditional expectation \( m_t \triangleq \mathbb{E}[Y_t|\mathcal{F}_{t-1}] \), we are interested in forecasting the unexpected return \( \epsilon_t = y_t - m_t \). In particular, we are interested in

\[
    h_t \triangleq \text{var}[Y_t|\mathcal{F}_{t-1}] \tag{1}
\]

Our goal in the succeeding will be to calibrate model parameters and extract i.i.d. standard variates,

\[
    z_t \triangleq \frac{\epsilon_t}{\sqrt{h_t}}
\]

for \( t = 1, 2, \ldots, T \) from the timeseries history.

The purpose of introducing the non-constant scale \( h_t \) is to capture as much of the conditional variance in the residual as possible rendering \( Z_t \) approximately i.i.d.. We can then use these “de-volatized” residuals in various fashions; for example to fit a copula, or we can use the updating equations to translate the random variable \( Z_{T+1} \) into returns and thence into a forecast for market prices.

\[
    Y_{T+1} = m_{T+1} + Z_{T+1} \sqrt{h_{T+1}} \tag{2}
\]

Engle and Ng compare and several existing and new models. We shall focus on one of their new models, NGARCH(1,1), which is specified by\(^1\)

\[
    h_t = \omega + \alpha (\epsilon_{t-1} + \gamma \sqrt{h_{t-1}})^2 + \beta h_{t-1} \tag{3}
\]

This discrete process has a long memory through the auto-regressive term. It also exhibits a potentially asymmetric response to shocks through the \( \gamma \).

\(^1\)Note the typo near the bottom of page 1770 of the reference, an extra “P”. It is correctly specified on page 1755.
3 Estimation

Given values for the conditional means, \( m_t \), (or an assumption that they are negligible), values for the parameters, and assumptions for \( \epsilon_0 \) and \( h_0 \), we can recursively evaluate \( h_t \) for \( t = 1, \ldots, T \).

If the model is correctly-specified, then the residuals should be independent, if not identical. If we assume that they are normal with mean zero and variance \( h_t \), then the likelihood function is

\[
L(\omega, \alpha, \beta, \gamma) = \prod_{t=1}^{T} \frac{e^{-\frac{\epsilon_t^2}{2h_t}}}{\sqrt{2\pi h_t}}
\]

Thus the joint MLE for the parameters is

\[
\arg\min \sum_{t=1}^{T} \log h_t + \frac{\epsilon_t^2}{h_t}
\]

This is termed the quasi-MLE if we are not, in fact, prepared to assume that the residuals are normal. In this case, it would be superior to replace the normal density in the likelihood function with a more appropriate density.

3.1 Variance Targeting

If your timeseries is sufficiently long, it can make sense to use the unconditional variance to eliminate one of the parameters. In particular, let

\[
\sigma^2 = \text{var}[Y_t] = \text{var}[\epsilon_t] = \mathbb{E}[\epsilon_t^2]
\]

We can solve for this to get

\[
\sigma^2 = \frac{\omega}{1 - \alpha(1 + \gamma^2) - \beta}
\]

and this result can be used to replace \( \omega \) in the specification, where the \( \sigma^2 \) is set to the sample variance of the returns, \( \hat{\sigma}^2 \).

3.2 Initialization

Since \( h_1 \) depends on \( \epsilon_0 \) and \( h_0 \), we need to either make assumptions about these values or else include them amongst the parameters. One approach is to assume that \( \epsilon_0 = 0 \) In this case, \( h_1 = \omega + \alpha \gamma^2 h_0 + \beta h_0 \). Further assuming \( h_0 = h_1 \) give the result

\[
\epsilon_0 \triangleq 0
\]

\[
h_0 \triangleq \frac{\omega}{1 - \alpha \gamma^2 - \beta}
\]

an alternate assumption, useful with variance targeting, is \( h_0 \triangleq \sigma^2 \), the unconditional variance.
4 Forecasting

Say we have estimated our model based on sampling period $\tilde{\tau}$ and are interested in an investment horizon $\tau \triangleq n\tilde{\tau}$ for $n \in \mathbb{N}$.

Let us assume, for practicality, that the conditional means in the future are in fact unconditional

$$E [Y_t | F_{t-1}] = E [Y_t | F_T] \quad \forall t > T$$

4.1 Scaling Expectation

The conditional expectation scales naturally.

$$E \left[ \sum_{i=1}^{n} Y_{T+i} \bigg| F_T \right] = \sum_{i=1}^{n} E [Y_{T+i} | F_T] = nm_{T+1}$$

4.2 Scaling Variance

Scaling the variance is a bit more involved.

$$\text{var} \left[ \sum_{i=1}^{n} Y_{T+i} \bigg| F_T \right] = \sum_{i=1}^{n} \text{var} [Y_{T+i} | F_T] = h_{T+1} + E h_{T+2} + E h_{T+3} + \cdots + E h_{T+n}$$

where all of the expectations are conditioned on $F_T$.

For the asymmetric model we have been working with,

$$E h_{T+2} = \omega + h_{T+1} \left( \beta + \alpha \left( 1 + \gamma^2 \right) \right) \triangleq \omega + h_{T+1}\phi$$

Expected single-period conditional variances further in the future are seen to follow the pattern

$$E h_i = \omega \left( 1 + \phi + \phi^2 + \cdots + \phi^{i-2} \right) + h\phi^{i-1}$$

Summing these up, we get

$$\text{var} \left[ \sum_{i=1}^{n} Y_{T+i} \bigg| F_T \right] = \frac{1 - \phi^n}{1 - \phi} \left( h_{T+1} - \sigma^2 \right) + n\sigma^2$$

where $\sigma^2 \triangleq \frac{\omega}{1-\phi}$ is the unconditional single-period variance.

We can see that the GARCH forecast for cumulative variance over $n$ periods is a natural blend of the short-term and the long-term forecasts.
References