Practical Topics in Optimization
MFM Practitioner Module: Risk & Asset Allocation

John Dodson

February 27, 2013
Outline

Rebalancing & Normality

Allocation implied prior & CAPM

Measuring Skewness

Black-Litterman & the NIW Prior

Michaud & the NIW

SeDuMi & Practical Robust Optimization
Rebalancing & Normality

The basic fact asset values cannot be negative\(^1\) means that portfolio managers must deal with an inherent skewness to \(\Psi_\alpha = \alpha' M\). This should be either modeled explicitly or handled implicitly by periodically rebalancing the portfolio.

Rebalancing

Say you rebalance your portfolio \(n\) times between \(T\) and \(T + \tau\). Then effectively you replace the skewed random variables \(M = a + BP_{T+\tau}\) for each asset by

\[
M_{\text{bal}} \triangleq \sum_{i=1}^{n} \tilde{M}_i, \quad \tilde{M}_i \sim \tilde{a} + BP_{T+\tau}/n \quad \forall i = 1, \ldots, n
\]

if we can assume i.i.d. inter-period returns.

If \(n\) is sufficiently large, the Central Limit Theorem comes in and \(M_{\text{bal}}\) is approximately normal.

\(^1\)This is termed “limited liability”.

---

Outline
- Rebalancing & Normality
- Allocation implied prior & CAPM
- Measuring Skewness
- Black-Litterman & the NIW Prior
- Michaud & the NIW
- SeDuMi & Practical Robust Optimization
Allocation implied prior & CAPM

For normal markets, \( M \sim \mathcal{N}(\mu, \Sigma) \), a linear objective, \( \Psi_\alpha = \alpha' M \), and exponential utility, \( u(\psi) = -e^{-\psi/\zeta} \), we know that if we are given

- \( \Sigma \triangleq \Sigma_0 \), \( \zeta \triangleq \zeta_0 \) (covariance, risk aversion), and
- \( \arg \max_\alpha \mathbb{E} u(\Psi_\alpha) \triangleq \alpha_0 \) (optimality)

then

\[
\mu = \mu_0 \triangleq \frac{1}{\zeta_0} \Sigma_0 \alpha_0
\]

In particular,

\[
\zeta_0 = \frac{\text{var } \Psi_{\alpha_0}}{\mathbb{E} \Psi_{\alpha_0}}
\]

and in general for any portfolio \( \alpha \),

\[
\mathbb{E} \Psi_\alpha = \mathbb{E} \Psi_{\alpha_0} \frac{\text{cov} (\Psi_\alpha, \Psi_{\alpha_0})}{\text{var } \Psi_{\alpha_0}}
\]

The analogy to CAPM is clear: such an investor only expects to be compensated for systematic risk.
Measuring Skewness

In the profit objective setting with lognormal asset prices, we have already seen the first two co-moments of the market vector \( M \),

\[
E M = p \left( e^\mu + \frac{1}{2} \text{diag} \Sigma - 1 \right)
\]

\[
\text{cov} M = (p + E M) (p + E M)' \left( e^\Sigma - 1 \right)
\]

We need the third co-moment to measure the skew of \( \Psi_\alpha \). This is a third-order tensor, whose \((j, k, l)\)-element is

\[
\text{CM}^{(3)}_{j, k, l} M = (p_j + E M_j) (p_k + E M_k) (p_l + E M_l)
\]

\[
\left( 2 + e^{\Sigma_{j, k} + \Sigma_{j, l} + \Sigma_{k, l}} - e^{\Sigma_{j, k}} - e^{\Sigma_{j, l}} - e^{\Sigma_{k, l}} \right)
\]

from which you can measure the third central moment of \( \Psi_\alpha = \alpha' M \) and convert to skew.
Black-Litterman & the NIW Prior

The structure of Black-Litterman “picks” and “views” is designed for normal markets. But the mathematics applies equally well for normal market invariants, even if the market vector maps are nonlinear.

Lognormal Markets
For example, say $M = a + BP_{T+\tau}$ with $P_{T+\tau} = p_T e^X$ and $X \sim \mathcal{N}(\mu, \Sigma)$ with conjugate NIW $(\mu, \Sigma)$ with hyperparameters $\lambda_0, \nu_0, \mu_0, \Sigma_0$ where $E \mu = \mu_0$ and $E \Sigma^{-1} = \Sigma_0^{-1}$.

Setting the Prior
We would use Black-Litterman to help choose $\mu_0$ and $\Sigma_0$, even though $X$ is log-returns. The “picks” and “views” are now based on linear combinations of log-returns rather than investible portfolios, but your analyst may be willing to have his forecasts interpreted in these terms.
Michaud & the NIW

If we are working in a Bayesian setting, we can use the posterior characterization as a more coherent way to implement Michaud resampling.

Resampling

Rather than simulating datasets and re-fitting market vector parameters *per* Michaud, we can simply draw from the posterior distribution, perhaps using Gibbs sampling.

NIW variates

Generating pseudo-random NIW draws is relatively easy (at least for integer $\nu$). In fact, since the Wishart is related to the Gamma, and the Gamma is related to the Chi-Squared, a simple technique exists to convert an $(\nu \times \text{dim } M)$ block of standard normal variates to a Wishart variate.
In the mean-variance setting, robustness with respect to the characterization of the mean $\mu$ of the market vector is more important than robustness with respect to the characterization of the variance $\Sigma$. Meucci poses the following for “practical robust optimization”

$$\alpha^*(\gamma_m, \gamma_e) = \arg \max_{\alpha: d'\alpha \leq c, \alpha \geq 0} \mu_0'\alpha - \gamma_m \sqrt{\alpha'\Sigma_0\alpha} - \gamma_e \sqrt{\alpha'T\alpha}$$

where $\mu_0 \triangleq \mathbb{E}\mu$, $T \triangleq \text{cov}\mu$ and $\Sigma_0 \triangleq \mathbb{E}\Sigma$.

**Convex Programming**

In the technical appendix, Meucci demonstrates that this problem can be expressed in the “self-dual minimization” form and shows how this can be implemented in MATLAB using the CVX package.

The optimal portfolio would solve $\max_{\gamma_m, \gamma_e} \mathcal{S}(\alpha^*(\gamma_m, \gamma_e))$