Risk & Asset Allocation
Case for Spring 2

John A. Dodson
January 29, 2014

This case previews of upcoming sessions in order to demonstrate the concept of an allocation-implied prior model. Let us examine the equilibrium allocation under exponential utility and normal markets.

Model

For a portfolio \( \alpha_0 + \alpha \) (\( \alpha_0 \) represents cash) with net asset value

\[
w = \alpha_0 + \alpha^\top p
\]

the gain/loss over \( \tau \) years would be

\[
\Psi = \alpha_0 r \tau + \alpha^\top M
\]

where the market vector is

\[
M = P - p
\]

with \( P \) the random variable for asset prices, including any cashflows, \( \tau > 0 \) years in the future and \( r \) the (simple-interest) return on cash.

Let us assume that the market vector is normal,

\[
M \sim \mathcal{N}(\mu_\tau, \Sigma_\tau)
\]

and that the preferences of the representative agent are described by exponential utility

\[
u(\psi) = -e^{-\frac{\psi}{\zeta}}
\]

with absolute risk aversion \( 1/\zeta > 0 \).

Let us consider the portfolios that satisfy a wealth constraint \( w^* \) and maximize expected utility.

\[
E u(\Psi) = e^{-\frac{\alpha_0 r \tau}{\zeta}} E e^{-\frac{\alpha^\top M}{\zeta}}
\]

\[
= e^{-\frac{w^* - \alpha_0 r \tau}{\zeta} - \frac{\alpha^\top \mu_\tau}{\zeta} + \frac{1}{2} \frac{\alpha^\top \Sigma_\tau \alpha}{\zeta}}
\]

So an optimal portfolio satisfies

\[
\alpha^* \in \arg \max_{\alpha} \alpha^\top (\mu - rp) - \frac{1}{2\zeta} \alpha^\top \Sigma \alpha
\]
If the covariance is positive-definite, $\Sigma > 0$ (which it would not be if cash were included in the market vector), the first-order condition on the optimal portfolio is

$$\mu = rp + \frac{1}{\zeta} \Sigma \alpha^*$$

(9)

This relationship, linking the market characterization to the investor utility and the optimal portfolio, can be the basis for an allocation-implied prior.

Notice that

$$E \Psi^* = w^* r \tau + \frac{1}{\zeta} \text{var} \Psi^*$$

(10)

and more generally that

$$E \Psi = w^* r \tau + \frac{1}{\zeta} \text{cov} (\Psi, \Psi^*)$$

(11)

$$= w^* r \tau + \frac{\text{cov} (\Psi, \Psi^*)}{\text{var} \Psi^*} (E \Psi^* - w^* r \tau)$$

(12)

This is more recognizable as

$$E \frac{\Psi}{w^* \tau} = r + \frac{\text{cov} (\Psi, \Psi^*)}{\text{var} \Psi^*} \left( E \frac{\Psi^*}{w^* \tau} - r \right)$$

(13)

where the coefficient is akin to “beta” in the capital asset pricing model.

Consider a portfolio consisting of a single share of the $i$-th stock.

$$\Psi = \frac{P_i}{p_i} - 1$$

(14)

Hence

$$E P_i = p_i (1 + \bar{R}_i) + \lambda \text{cor} (P_i, \Psi^*) \sqrt{\tau \text{var} P_i}$$

(15)

where

$$\lambda \equiv \frac{\sqrt{\alpha^* \text{cov} \Psi^*}}{\zeta}$$

(16)

with dimensions $yr^{-1/2}$ is termed the “market price of risk” and notably depends on neither the asset nor the investment horizon.

In particular, the expected value of the (simple) return on the $i$-th asset is

$$\bar{R}_i \equiv r + \lambda \text{cor} (P_i, \Psi^*) \sqrt{\frac{\text{var} P_i}{p_i^2 \tau}}$$

(17)

whereby

$$E P_i = p_i (1 + \bar{R}_i \tau)$$

(18)