Risk & Asset Allocation (Spring)
Exercise for Week 4

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Let us try out the two-step approach to determine the optimal portfolio under a single affine constraint with objective linear in the market vector and index of satisfaction equal to the Cornish-Fisher expansion of the 95% expected shortfall.

For an objective defined by \( \Psi_\alpha = \alpha' M \) and an affine constraint defined by \( d' \alpha = c \), the analytic solution to the optimal mean-variance portfolio satisfies

\[
\alpha(\beta) = (1 - \beta) \alpha_{MV} + \beta \alpha_{SR}
\]

for \( \beta > 0 \), where

\[
\alpha_{MV} = \frac{c}{d' (\text{Cov}M)^{-1} d},
\]

\[
\alpha_{SR} = \frac{c}{d' (\text{Cov}M)^{-1} E M}
\]

We need to determine the level of \( \beta \) that maximizes the index of satisfaction, which we will take to be

\[
S(\alpha) = E \Psi_\alpha + \sqrt{\text{Var} \Psi_\alpha} \left( I \left[ \phi \Phi^{-1} \right] + \frac{1}{6} \left( I \left[ \phi (\Phi^{-1})^2 \right] - 1 \right) \text{Skew} \Psi_\alpha \right)
\]

based on the Cornish-Fisher expansion, where \( \Phi \) is the CDF of a standard normal random variable and \( \phi \) is the spectrum for \( ES_{0.95} \).

Since we can assume that the skewness of \( M \) is negligible, the skewness of \( \Psi_\alpha \) is also negligible. Furthermore, we can evaluate the integral in the expansion numerically.

\[
I \left[ \phi \Phi^{-1} \right] = \int_0^{0.05} \frac{\sqrt{2} \text{erf}^{-1}(2p - 1)}{0.05} \, dp \approx -2.0627 \ldots
\]

Let us assign \( z_{0.95} = 2.0627 \ldots \), so the integral above is \(-z_{0.95}\). The satisfaction is

\[
S(\alpha) = \alpha' EM - z_{0.95} \sqrt{\alpha' (\text{Cov}M) \alpha}
\]

Substituting in (1), we get that the optimal value for \( \beta \) is

\[
\beta^* = \arg \max_{\beta > 0} \left( (1 - \beta) \alpha_{MV} + \beta \alpha_{SR} \right)' EM
\]

\[
- z_{0.95} \sqrt{((1 - \beta) \alpha_{MV} + \beta \alpha_{SR})' (\text{Cov}M) ((1 - \beta) \alpha_{MV} + \beta \alpha_{SR})}
\]
From manipulation of the first-order condition, recognizing that

\[ \text{Cov}(\Psi_{\alpha_{MV}}, \Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}) = 0 \]

we can determine that the solution is

\[
\beta^* = \begin{cases} 
0 & \gamma \leq 0 \\
\sqrt{\frac{\text{Var}\Psi_{\alpha_{MV}}}{\text{Var}(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}}) \left( \frac{z_{0.95}}{\gamma} \right)^2 - 1}} & 0 < \gamma < z_{0.95} \\
\infty & \gamma \geq z_{0.95}
\end{cases}
\]

where

\[
\gamma = \frac{\mathbb{E}(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}})}{\sqrt{\text{Var}(\Psi_{\alpha_{SR}} - \Psi_{\alpha_{MV}})}}
\]

is the market price for risk.

In conclusion, the optimal portfolio is \( \alpha(\beta^*) \).