Let us consider the expected shortfall index of satisfaction for a very simple portfolio: $\alpha$ shares in an asset whose value today is $p > 0$ and whose horizon value $P$ is lognormal.

Let us assume that the objective measure is profit; therefore in Meucci’s notation, we have

$$
\Psi_\alpha = \alpha M
= \alpha (P - p)
= \alpha (g(X) - p)
= \alpha p(e^X - 1)
$$

where the invariant total return is normal $X \sim \mathcal{N}(\mu, \Sigma)$ with mean $\mu$ and variance $\Sigma > 0$. The index of satisfaction is

$$
S(\alpha) = \frac{1}{1 - c} \int_0^{1-c} Q_{\Psi_\alpha}(q) \, dq
$$

for confidence level $c < 1$ in terms of the quantile function for the objective value.

## 1 Exact Version

In this simple situation, we can actually calculate a relatively simple expression for the value of index of satisfaction. It will be useful to compare this below with the approximate value we get from the Cornish-Fisher expansion.

We proceed to evaluate the exact version by considering the CDF of the objective.

$$
F_{\Psi_\alpha}(z) = P\{\Psi_\alpha < z\}
= P\{\alpha p(e^X - 1) < z\}
= P\left\{X \, \text{sgn} \alpha < \log\left(1 + \frac{z}{\alpha p}\right) \, \text{sgn} \alpha\right\}
= P\left\{\frac{X - \mu}{\sqrt{\Sigma}} \, \text{sgn} \alpha < \frac{\log\left(1 + \frac{z}{\alpha p}\right) - \mu}{\sqrt{\Sigma}} \, \text{sgn} \alpha\right\}
= \Phi\left(\frac{\log\left(1 + \frac{z}{\alpha p}\right) - \mu}{\sqrt{\Sigma} \, \text{sgn} \alpha}\right)
$$

1
where $\Phi(\cdot)$ is the CDF of a standard normal.

The quantile, which is the inverse of the CDF, is therefore

$$Q_{\Psi(\alpha)}(q) = \alpha p \left( e^{\mu + \mathrm{sgn} \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right)$$

So can proceed to evaluate the index of satisfaction.

$$S(\alpha) = \frac{1}{1 - c} \int_0^{1-c} \alpha p \left( e^{\mu + \mathrm{sgn} \alpha \sqrt{\Sigma} \Phi^{-1}(q)} - 1 \right) dq$$

$$= \alpha p \left( \frac{1}{1 - c} \int_0^{1-c} e^{\mu + \mathrm{sgn} \alpha \sqrt{\Sigma} \Phi^{-1}(q)} dq - 1 \right)$$

$$= \alpha p \left( \frac{1}{1 - c} \int_{-\infty}^{\Phi^{-1}(1-c)} e^{\mu + \mathrm{sgn} \alpha \sqrt{\Sigma} \phi(z)} dz - 1 \right)$$

where the last line is achieved by the change of variable $z = \Phi^{-1}(q)$ and $\phi(z) = \Phi'(z)$ is the density of a standard normal.

Since

$$e^{\mu + \mathrm{sgn} \alpha \sqrt{\Sigma} \phi(z)} = e^{\mu + \frac{1}{2} \Sigma \phi \left( z - \mathrm{sgn} \alpha \sqrt{\Sigma} \right)}$$

we have the final result,

$$S(\alpha) = \alpha p \left( e^{\mu + \frac{1}{2} \Sigma} \frac{1}{1 - c} \Phi \left( \Phi^{-1}(1-c) - \mathrm{sgn} \alpha \sqrt{\Sigma} \right) - 1 \right)$$

(1)

2 Short Horizon Approximation

For short horizons, the mean and variance of the total return invariant are small. To lowest order, the exact result in (1) can be approximated by

$$S(\alpha) \approx \alpha p \left( \mu - \mathrm{sgn} \alpha \frac{\phi \left( \Phi^{-1}(1-c) \right) \sqrt{\Sigma}}{1 - c} \right)$$

(2)

Let us spend a moment interpreting this. An investor will be more satisfied to be long ($\alpha > 0$) if the asset has a positive expected return ($\mu > 0$), and short ($\alpha < 0$) if the asset has a negative expected return ($\mu < 0$). In contrast, positive variance diminishes satisfaction for any non-zero position.

This all seems quite reasonable for a rational index of satisfaction.

3 Cornish-Fisher Approximation

It is unusual to have a simple analytic expression for the expected shortfall such as (1). This is why the Cornish-Fisher expansion can be useful in practice. In order to use this, we need several low central moments for the objective $\Psi_{\alpha}$. In a Delta-Gamma setting, we can replace the objective by the quadratic

$$\Psi_{\alpha} = \alpha p \left( e^{X} - 1 \right) \approx \alpha p \left( X + \frac{1}{2} X^2 \right)$$
hence $\Theta_\alpha = 0$, $\Delta_\alpha = \alpha p$, and $\Gamma_\alpha = \alpha p$. Let us define a new objective to represent this approximation.

$$\Xi_\alpha = \alpha p \left( X + \frac{1}{2} X^2 \right)$$

Is is straight-forward to work out that the first several central moments of this are

$$\begin{align*}
\mathbb{E}(\Xi_\alpha) &= \alpha p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) \\
\text{Sd}(\Xi_\alpha) &= |\alpha| p \sqrt{\Sigma} \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} \\
\text{Sk}(\Xi_\alpha) &= 3 \text{sgn} \alpha \sqrt{\Sigma} \frac{(1 + \mu)^2 + \frac{1}{2} \Sigma}{(1 + \mu)^2 + \frac{1}{2} \Sigma}^{3/2}
\end{align*}$$

The third-order Cornish-Fisher expansion for expected shortfall in general is

$$S(\alpha) \approx \mathbb{E}(\Xi_\alpha) + \text{Sd}(\Xi_\alpha) \left( z_1 + \frac{1}{6} z_2 - \text{Sk}(\Xi_\alpha) \right)$$

with coefficients

$$\begin{align*}
z_1 &= \frac{1}{1 - c} \int_0^{1-c} \Phi^{-1}(q) \, dq = -\frac{\phi \left( \Phi^{-1}(1-c) \right)}{1 - c} \\
z_2 &= \frac{1}{1 - c} \int_0^{1-c} \Phi^{-1}(q)^2 \, dq = 1 - \frac{\phi \left( \Phi^{-1}(1-c) \right)}{1 - c} \Phi^{-1}(1-c)
\end{align*}$$

depending on the confidence level $c < 1$.^1

Putting this together, we get a third expression for the index of satisfaction.

$$S(\alpha) \approx \alpha p \left( \mu + \frac{1}{2} \mu^2 + \frac{1}{2} \Sigma \right) - |\alpha| p \frac{\phi \left( \Phi^{-1}(1-c) \right)}{1 - c} \sqrt{\Sigma} \cdot \left( \sqrt{(1 + \mu)^2 + \frac{1}{2} \Sigma} + \frac{1}{2} \text{sgn} \alpha \frac{(1 + \mu)^2 + \frac{1}{2} \Sigma \Phi^{-1}(1-c) \sqrt{\Sigma}}{(1 + \mu)^2 + \frac{1}{2} \Sigma} \right) \quad (3)$$

This result agrees with (2) to lowest order in $\mu$ and $\Sigma$.

^1The trick to these integrals is to realize that $\phi'(z) = -z \phi(z)$.
4 Modeling Default

Our horizon asset value $P$ is bounded below by zero in this set-up. But if this is a model for a financial asset, we probably need to consider how the possibility of default would change the value of the expected shortfall. An amendment to the market model to consider is

$$\Psi'_\alpha = \alpha p (Ye^X - 1)$$

where $X \sim \mathcal{N}(\mu, \Sigma)$ as before\(^2\), but now we add an independent default indicator $Y \sim \text{Bern}(1 - h)$ for default probability $h$.

\(^2\text{Since we cannot observe default events in the historical record for the total return, there is no reason to alter the objective model for the invariant.}\)