Introduction

Treasury nominal\(^1\) bonds pay coupons at a fixed rate in equal installments twice per year. The usual approach to valuing bonds is based on the net present value of these cashflows. The basis for this is a discounting function, which could be arbitrarily complicated. Faced with estimation, a dimension reduction is called for. There are several popular functional forms for the discounting function. We will consider the following:

\[
d(T - t; \beta) = \exp \left\{ -\beta_2 (T - t) + \beta_3 \frac{1 - e^{-\beta_1 (T - t)}}{\beta_1} - \left( \frac{1 - e^{-\beta_4 (T - t)}}{\beta_1} \right)^2 \right\}
\]  

(1)

If a new par bond with maturity date \(T\) has yield (equal to coupon) \(y(t, T)\) on date \(t\), then

\[
1 = d(T - t; \beta(t)) + \frac{1}{2} y(t, T) \sum_{i=1}^{2(T-t)} d(i/2; \beta(t))
\]

(2)

can be used to define—or at least to fit—parameters \(\beta(t)\) to this model. One approach to this is (non-linear) least-squares

\[
\hat{\beta}(t) \triangleq \arg \min_{\beta} \sum_{j} \left( \frac{1 - d(T_j - t; \beta)}{\frac{1}{2} \sum_{i=1}^{2(T_j-t)} d(i/2; \beta)} - y(t, T_j) \right)^2
\]

where \(j\) indexes the yields to be fit.

Problem

A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, January 28.

1. Using the Federal Reserve’s H.15 statistical release, available here:

   [http://www.federalreserve.gov/releases/h15/current/default.htm](http://www.federalreserve.gov/releases/h15/current/default.htm)

   fit parameters \(\hat{\beta}(t)\) to \(T_j - t = 1-, 2-, 3-, 5-, 7-, 10-, 20-,\) and 30-year constant maturity Treasury nominal yields \(y(t, T_j)\) for \(t = January 14.\) Keep in mind that these yields are quoted in percent.  (6 points)

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\(^1\)“nominal” in contrast to “discount”, “inflation-protected”, or “registered interest and principal”
2. Make a table of the yields you used and the fitted yields you got with the parameters from above (based on (2)). If you are successful, these should be within about 5 basis points (5.E-4/yr) for each maturity. (4 points)

Solution

Below is the MATLAB I wrote to identify the parameters for the January 14 nominal Treasury yields.

```
>> tenor=[1 2 3 5 7 10 20 30]; % constant maturity Treasuries tenors
>> yield=[0.18 0.51 0.83 1.33 1.62 1.86 2.20 2.47]/100; % from Fed H.15 for 2015/01/14
>> disc=@(T,b)exp(-b(2)*T+b(3)*(1-exp(-b(1)*T))/b(1)-(b(4)*(1-exp(-b(1)*T))/b(1)).^2);
>> yld=@(T,b)arrayfun(@(T)2*(1-disc(T,b))/sum(disc(.5:.5:T,b)),T);
>> obj=@(b)sum(arrayfun(@(T,cpn)(yld(T,b)-cpn).^2,tenor,yield));
>> b0=fminsearch(obj,[.05 .05 .05 .05]);

The results I got for the parameters are here

Table 1: Least-squares (yield) parameters for January 14.

<table>
<thead>
<tr>
<th>β</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>β₁</td>
<td>0.2136 / yr</td>
</tr>
<tr>
<td>β₂</td>
<td>0.0283 / yr</td>
</tr>
<tr>
<td>β₃</td>
<td>0.0318 / yr</td>
</tr>
<tr>
<td>β₄</td>
<td>0.0473 / yr</td>
</tr>
</tbody>
</table>

and the fitted yields are here

Table 2: Fit and residuals for January 14.

<table>
<thead>
<tr>
<th>tenor</th>
<th>data</th>
<th>fit</th>
<th>error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 y</td>
<td>0.0018 / yr</td>
<td>0.0016 / yr</td>
<td>-0.0002 / yr</td>
</tr>
<tr>
<td>2 y</td>
<td>0.0051 / yr</td>
<td>0.0055 / yr</td>
<td>+0.0003 / yr</td>
</tr>
<tr>
<td>3 y</td>
<td>0.0083 / yr</td>
<td>0.0085 / yr</td>
<td>+0.0002 / yr</td>
</tr>
<tr>
<td>5 y</td>
<td>0.0133 / yr</td>
<td>0.0130 / yr</td>
<td>-0.0003 / yr</td>
</tr>
<tr>
<td>7 y</td>
<td>0.0162 / yr</td>
<td>0.0159 / yr</td>
<td>-0.0003 / yr</td>
</tr>
<tr>
<td>10 y</td>
<td>0.0186 / yr</td>
<td>0.0187 / yr</td>
<td>+0.0001 / yr</td>
</tr>
<tr>
<td>20 y</td>
<td>0.0220 / yr</td>
<td>0.0227 / yr</td>
<td>+0.0007 / yr</td>
</tr>
<tr>
<td>30 y</td>
<td>0.0247 / yr</td>
<td>0.0242 / yr</td>
<td>-0.0005 / yr</td>
</tr>
</tbody>
</table>

In order to plot a yield curve, I need to invert (2). I also need to extend its range to non-integer tenors. The usual way to do this is to introduce the concept of “clean price”, whereby “accrued interest” on the current coupon is excluded. The net present value of a par bond is then par plus accrued.

\[
1 + y(t, T) \left( \frac{1}{2} \left[ 2(T - t) \right] - T + t \right) = d(T - t; \beta(t)) + \frac{1}{2} y(t, T) \sum_{i=0}^{[2(T-t)]-1} d(T - i/2 - t; \beta(t)) \] (2')

\(y(t, T)\) defined implicitly here is continuous in \(T\) (as long as \(d(T - t)\) is for \(T > t\) and \(\lim_{T \to t} d(T - t) = 1\)) because both sides gap by \(\frac{1}{2} y(t, T)\) when \(T - t\) crosses a half-integer, in accordance with a coupon payment.
Discussion

While described as strictly empirical, this model has a basis in theory. If the coefficients are observed to have a certain dynamic, namely $\beta_1, \beta_2, \beta_4$ constant and $\beta_3(t)$ an Ornstein-Uhlenbeck process, this model is equivalent to the Vasicek (1977) single-factor gaussian short-rate model:

$$d(T - t) = E^Q \left[ e^{-\int_t^T r_s \, ds} \bigg| \mathcal{F}_t \right]$$

where

$$dr_t = (2\beta_2^2 + \beta_1 \beta_2 - \beta_1 r_t) \, dt + 2\beta_4 \sqrt{\beta_1} \, dW$$

$$r_t = \beta_2 - \beta_3(t)$$

with $W$ a standard brownian motion under $Q$. 

Figure 1: Nominal Treasuries par curve for January 14, 2015