Risk & Asset Allocation (Spring)
Homework for Week 3

John A. Dodson
February 4, 2015

A solution to this problem is due at the beginning of the next session, which is 5:30 PM on Wednesday, February 11.

Problem

Assume that the value for a portfolio over some analysis horizon is a random variable, $\Psi$, with a Generalized Pareto left tail partially defined by the distribution function $F_{\Psi}(\psi) \approx \begin{cases} \theta \left( 1 + \frac{\psi \eta - \psi \beta}{\beta} \right)^{-1/\xi} & \psi \leq \eta \\ ? & \text{otherwise} \end{cases}$

with tail parameter $0 < \xi < 1$, scale parameter $\beta > 0$, threshold parameter $\eta$, and tail mass $0 < \theta < 1$.

- Provide expressions for the value-at-risk (2 points) and expected shortfall (3 points) at confidence level $c$ with $1 - \theta \leq c < 1$ in terms of the parameters of this partial characterization.

- Let’s say we add some cash to the portfolio, so that the new horizon value is $\Psi' \triangleq \Psi + \psi_b$ for some fixed $\psi_b \geq 0$. What is the new distribution function $F_{\Psi'}(\cdot)$ (3 points)? Provide expressions for the value-at-risk (1 point) and expected shortfall for the same confidence as above for this new portfolio. Are these metrics translation-invariant (1 point)?

Solution

Value-at-risk is based on the quantile function, which is the inverse of the distribution function. For a sufficiently high confidence level, we can invert $F_{\Psi}(\cdot)$ to get

$$\text{VaR}_c \triangleq Q_{\Psi}(1-c) = \eta - \frac{\beta}{\xi} \left( \left( \frac{\theta}{1-c} \right)^{\xi} - 1 \right) \quad (1)$$

To evaluate the expected shortfall, we can use the definition

$$\text{ES}_c \triangleq \frac{1}{1-c} \int_0^{1-c} Q_{\Psi}(p) \, dp$$
to get

\[ \text{ES}_c = \eta - \frac{\beta}{\xi} \left( \frac{1}{1 - \xi} \left( \frac{\theta}{1 - c} \right)^\xi - 1 \right) \]  

(2)

As an aside, McNeil et al. points out that for high confidence levels, \( \eta + \frac{\beta}{\xi} \) is negligible; so

\[ \lim_{c \uparrow 1} \frac{\text{ES}_c}{\text{VaR}_c} = \frac{1}{1 - \xi} \]

This is a nice way to think about the interpretation of the tail parameter, \( \xi \). Expected shortfall is larger than value-at-risk; this relationship tells us by how much. Also, a normal tail has \( \xi = 0 \), so this result highlights how expected shortfall and value-at-risk diverge for non-normal tails.

With cash included, the new distribution function come from the old simply by substituting the argument \( \psi \mapsto \psi - \psi_b \).

\[ F_{\Psi'}(\psi) \approx \begin{cases} \theta \left( 1 + \frac{\eta - \psi + \psi_b}{\beta} \right)^{-1/\xi} & \psi \leq \eta + \psi_b \\ ? & \text{otherwise} \end{cases} \]

Notice that this is equivalent to

\[ F_{\Psi'}(\psi) \approx \begin{cases} \theta \left( 1 + \frac{\eta' - \psi}{\beta} \right)^{-1/\xi} & \psi \leq \eta' \\ ? & \text{otherwise} \end{cases} \]

where \( \eta' \triangleq \eta + \psi_b \). That is, the addition of cash is reflected simply by shifting the threshold parameter.

Since equations (1) and (2) for value-at-risk and expected shortfall are additive in \( \eta \), these metrics as indexes of satisfaction are translation invariant.