A solution to this problem is due at the beginning of the last(!) session, which is 5:30 PM on Wednesday, February 25.

Consider a broad investment universe and a normal market vector. Assume: all initial asset prices are one; under the objective probability measure each pair-wise correlation is 0.6; and each component’s mean and variance is +0.01 and (0.1)^2 respectively.

The manager’s subjective probability measure is based on a view with confidence 0.3 that one particular market vector component (e.g. the first) will turn out to be +0.05.

- What is the mean and variance of that component under the Black-Litterman subjective probability measure? (2 points)

- Under the manager’s subjective probability measure, what fraction of the SR portfolio’s initial value should be invested in this component? (8 points)

- Express the answers exactly in terms of the common marginal mean \( \mu_0 \in \mathbb{R} \), variance \( \sigma_0^2 > 0 \), and (Gaussian) copula parameter \( 0 < \rho < 1 \) characterizing the objective probability measure, and the view component \( v_1 \in \mathbb{R} \) and confidence \( 0 < c < 1 \). (2 points)

**Solution**

The picks matrix is simply

\[ P = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix} \]

So \( P \Sigma P' \) is just the scalar \( \Sigma_{11} = \sigma_0^2 \) and

\[
\Sigma P' = \begin{pmatrix}
\sigma_0^2 \\
\sigma_0^2 \rho \\
\sigma_0^2 \\
\vdots
\end{pmatrix}
\]

The Black-Litterman market vector mean is

\[
\mu_{BL} = \mu + c \Sigma P' \left( P \Sigma P' \right)^{-1} (v - P \mu) = \begin{pmatrix}
(1 - c)\mu_0 + cv_1 \\
(1 - c\rho)\mu_0 + c\rho v_1 \\
(1 - c\rho)\mu_0 + c\rho v_1 \\
\vdots
\end{pmatrix}
\]
Notice that the marginal variances factor out.

- To answer the first question, we see that \([\mu_{BL}]_1 = (1 - c)[\mu]_1 + cv_1 = +0.022\)

To evaluate the \(\alpha_{SR}\) portfolio for the second question, we need first to evaluate

\[
\Sigma_{BL} = \Sigma - c\Sigma P' \left( P\Sigma P' \right)^{-1} P\Sigma
\]

Since \(\Sigma\) is symmetric, \(\Sigma P' = (P\Sigma)'\). Thus we can arrive at

\[
\Sigma_{BL} = \sigma_0^2 \begin{pmatrix}
1 - c & \rho - cp & \rho - cp^2 & \cdots \\
\rho - cp & 1 - c\rho^2 & \rho - c\rho^2 & \cdots \\
\rho - cp & \rho - c\rho & 1 - c\rho & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

- In particular, \([\Sigma_{BL}]_{11} = (1 - c) [\Sigma]_{11} = 0.007\)

The inverse of this is not necessarily apparent, but turns out to be

\[
\Sigma_{BL}^{-1} = \frac{1}{\sigma_0^2(1 - \rho)} \begin{pmatrix}
\frac{1 - \rho}{1 - c} + \frac{(n - 1)^2\rho}{1 + (n - 1)\rho} & -\frac{\rho}{1 + (n - 1)\rho} & -\frac{\rho}{1 + (n - 1)\rho} & \cdots \\
-\frac{\rho}{1 + (n - 1)\rho} & 1 - \frac{(n - 1)^2\rho}{1 + (n - 1)\rho} & -\frac{(n - 1)^2\rho}{1 + (n - 1)\rho} & \cdots \\
-\frac{\rho}{1 + (n - 1)\rho} & -\frac{(n - 1)^2\rho}{1 + (n - 1)\rho} & 1 - \frac{(n - 1)^2\rho}{1 + (n - 1)\rho} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(2)

where \(n = \dim M\) is the number of assets in the investment universe. The proof of this comes from multiplying out \(\Sigma_{BL}^{-1}\) and \(\Sigma_{BL}\).

Assuming the initial price vector \(p\) is one (or at least proportional to one), the SR portfolio allocation is proportional to

\[
\alpha_{SR} \propto \Sigma_{BL}^{-1}\mu_{BL} = \frac{1}{\sigma_0^2} \begin{pmatrix}
\frac{v_1}{1 - c} + \frac{\mu_0}{1 + (n - 1)\rho} \\
\frac{\mu_0}{1 + (n - 1)\rho} \\
\vdots
\end{pmatrix}
\]

\[
[p]_1 [\alpha_{SR}]_1 \propto \frac{v_1}{1 - c} + \frac{\mu_0}{1 + (n - 1)\rho}
\]

\[
p' \alpha_{SR} \propto \frac{v_1}{1 - c} + \frac{\mu_0}{1 + (n - 1)\rho}
\]

For a sufficiently broad investment universe, this limits to

\[
\lim_{n \to \infty} \frac{[p]_1 [\alpha_{SR}]_1}{p' \alpha_{SR}} = \frac{1}{1 + \frac{\mu_0}{v_1}}\frac{1 - c}{1 - c\rho}
\]

(3)

- For the parameters given in the problem, this limit evaluates to about 56%. The other 44% should be allocated equally to the remaining assets.

You should get a numerical value close to the exact result for any model with at least twenty assets in the investment universe.