Practical Topics in Optimization

MFM Practitioner Module: Risk & Asset Allocation

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SeDuMi & Practical Robust Optimization
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GARCH models explicitly violate the conditions for the “square-root rule” scaling. Say we have estimated our model based on sampling period $\tilde{\tau}$ and our investment horizon is $\tau \triangleq n\tilde{\tau}$ for $n \in \mathbb{N}$. Let us assume (counterfactually, but for practicality) that the conditional means in the future are in fact unconditional

$$E[Y_t|\mathcal{F}_{t-1}] = E[Y_t|\mathcal{F}_T] \quad \forall t > T$$

The conditional expectation scales naturally.

$$E \left[ \sum_{i=1}^{n} Y_{T+i} \middle| \mathcal{F}_T \right] = \sum_{i=1}^{n} E[Y_{T+i}|\mathcal{F}_T] = nm_{T+1}$$
Multiperiod Forecasting

Scaling the variance is more subtle.

\[
\text{var} \left[ \sum_{i=1}^{n} Y_{T+i} \bigg| F_T \right] = \sum_{i=1}^{n} \text{var} [ Y_{T+i} \big| F_T ] = h_{T+1} + E h_{T+2} + E h_{T+3} + \cdots + E h_{T+n}
\]

where all of the expectations are conditioned on \( F_T \).

For the asymmetric model we have been working with,

\[
E h_{T+2} = \omega + h_{T+1} \left( \beta + \frac{1}{2} (\alpha^+ + \alpha^-) \right) = \omega + h_{T+1} \phi
\]

Further in the future we get the following the pattern

\[
E h_{T+i} = \omega \left( 1 + \phi + \phi^2 + \cdots + \phi^{i-2} \right) + h_{T+1} \phi^{i-1}
\]
**Multiperiod Forecasting**

Summing these up, we get

$$\text{var} \left[ \sum_{i=1}^{n} Y_{T+i} \mid \mathcal{F}_T \right] = \frac{1 - \phi^n}{1 - \phi} (h_{T+1} - \sigma^2) + n\sigma^2$$

where $\sigma^2 \triangleq \frac{\omega}{1-\phi}$ is the unconditional single-period variance.

We can see that the GARCH forecast for cumulative variance over $n$ periods is a blend of the short-term and the long-term forecasts.

- For $\phi \approx 1$ (unit root), scale the single-period forecast.
- For $\phi \ll 1$ and $n \gg 1$, scale the unconditional variance.
Multiperiod Forecasting

Figure: Forecasting variance example
Rebalancing & Normality

The basic fact asset values cannot be negative means that portfolio managers must deal with an inherent skewness to $\Psi_{\alpha} = \alpha' M$. This should be either modeled explicitly or handled implicitly by periodically rebalancing the portfolio.

Rebalancing

Say you rebalance your portfolio $n$ times between $T$ and $T + \tau$. Then effectively you replace the skewed random variables $M = a + BP_{T+\tau}$ for each asset by

$$M_{\text{bal}} \triangleq \sum_{i=1}^{n} \tilde{M}_i, \quad \tilde{M}_i \sim \tilde{a} + BP_{T+\tau}/n \quad \forall i = 1, \ldots, n$$

if we can assume i.i.d. inter-period returns.

If $n$ is sufficiently large, the Central Limit Theorem comes in and $M_{\text{bal}}$ is approximately normal.

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$^1$This is termed “limited liability”.
Measuring Skewness

In the profit objective setting with lognormal asset prices, we have already seen the first two co-moments of the market vector $M$,

$$
E M = p \left( e^{\mu} \frac{1}{2} \text{diag} \Sigma - 1 \right)
$$

$$
cov M = (p + E M) (p + E M)' \left( e^\Sigma - 1 \right)
$$

We need the third co-moment to measure the skew of $\Psi_\alpha$. This is a third-order tensor, whose $(j, k, l)$-element is

$$
CM^{(3)}_{j,k,l} M = (p_j + E M_j) (p_k + E M_k) (p_l + E M_l)
$$

$$
\left( 2 + e^{\Sigma_{j,k} + \Sigma_{j,l} + \Sigma_{k,l}} - e^{\Sigma_{j,k}} - e^{\Sigma_{j,l}} - e^{\Sigma_{k,l}} \right)
$$

from which you can measure the third central moment of $\Psi_\alpha = \alpha' M$ and convert to skew.
Black-Litterman & the NIW Prior

The structure of Black-Litterman “picks” and “views” is designed for normal markets. But the mathematics applies equally well for normal market invariants, even if the market vector maps are nonlinear.

Lognormal Markets

For example, say $M = a + BP_{T+\tau}$ with $P_{T+\tau} = p_T e^X$ and $X \sim \mathcal{N}(\mu, \Sigma)$ with conjugate NIW $(\mu, \Sigma)$ with hyperparameters $\lambda_0, \nu_0, \mu_0, \Sigma_0$ where $E\mu = \mu_0$ and $E\Sigma^{-1} = \Sigma_0^{-1}$.

Setting the Prior

We would use Black-Litterman to help choose $\mu_0$ and $\Sigma_0$, even though $X$ is log-returns. The “picks” and “views” are now based on linear combinations of log-returns rather than investible portfolios, but your analyst may be indifferent to this distinction.
Michaud & the NIW

If we are working in a Bayesian setting, we can use the posterior characterization as a more coherent way to implement Michaud resampling.

Resampling

Rather than simulating datasets and re-fitting market vector parameters \emph{per} Michaud, we can simply draw from the posterior distribution, perhaps using Gibbs sampling.

NIW variates

Generating pseudo-random NIW draws is relatively easy (at least for integer $\nu$). In fact, since the Wishart is related to the Gamma, and the Gamma is related to the Chi-Squared, a simple technique exists to convert an $(\nu \times \text{dim } M)$ block of standard normal variates to a Wishart variate.
In the mean-variance setting, robustness with respect to the characterization of the mean $\mu$ of the market vector is more important than robustness with respect to the characterization of the variance $\Sigma$. Meucci poses the following for “practical robust optimization”

$$\alpha^*(\gamma_m, \gamma_e) = \arg \max_{\alpha: d'\alpha \leq c} \mu'_0 \alpha - \gamma_m \sqrt{\alpha' \Sigma_0 \alpha} - \gamma_e \sqrt{\alpha' T \alpha}$$

where $\mu_0 \triangleq \mathbb{E} \mu$, $T \triangleq \text{cov} \mu$ and $\Sigma_0 \triangleq \mathbb{E} \Sigma$.

**Convex Programming**

In the technical appendix, Meucci demonstrates that this problem can be expressed in the “self-dual minimization” form and shows how this can be implemented in MATLAB using the CVX package.

The optimal portfolio would solve $\max_{\gamma_m, \gamma_e} S(\alpha^*(\gamma_m, \gamma_e))$