Quantitative Risk Management
Homework for Week 2

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Problems

Solutions to these problems are due at the beginning of the next session, which is 5:30 PM on Wednesday, September 20.

Implied distribution

The Black-Scholes-Merton formula for (European-style) options values gives us a correspondence between option premiums and implied volatilities. For a call,

\[ c(K) = d \left( F \Phi \left( \frac{\log (F/K) \sigma(K)}{\sigma(K)} + \frac{1}{2} \sigma(K) \right) - K \Phi \left( \frac{\log (F/K)}{\sigma(K)} - \frac{1}{2} \sigma(K) \right) \right) \]

where \( d \) is the discount to expiration, \( F \) is the corresponding forward price of the underlying, \( K \) is the strike price, \( \sigma(K) \) is the implied standard deviation of the log of the underlying price at expiration under the BSM model, and \( \Phi(\cdot) \) is the standard normal (cumulative) distribution function.

For a one-year option, \( \sigma(K) \) corresponds to the Black volatility for the (European-style put or call) option with strike \( K \). More generally \( \sigma(K) \) can be scaled by the inverse square root of the option tenor to derive the corresponding Black implied volatility rate.

If \( \sigma(\cdot) \) is constant for all strike prices, the implied distribution of the underlying price at expiration is log-normal, which comes from the BSM assumption about the stochastic process for the underlying price. If it is not constant, the implied distribution is not log-normal\(^1\).

In the case of options on Apple Inc. common equity for expiration 9/21/2018, \( \sigma(\cdot) \) is not constant. In fact at the close of the regular session on 9/12/2017 it was approximately

\[ \sigma(K) \approx \sqrt{1.30 \times 10^{-6} + \sqrt{2.65 \times 10^{-3} + 1.93 \times 10^{-2} \log^2 (K/F)}} \]

where \( F \approx 161.11 \) and \( d \approx 0.99454 \).

Since in general

\[ c''(K) = d f(K) \quad \forall \ K > 0 \]

\(^1\)Note that while the volatility curve is sufficient to describe the (risk-neutral) distribution of the terminal value of the underlying price, it is not sufficient to describe the (risk-neutral) stochastic process for the underlying price. Even the “volatility surface”, the implied volatility for each strike and tenor, is generally insufficient.
Figure 1: Implied BSM standard deviation for log AAPL on 9/21/2018 as of 9/12/2017 as a function of strike price.

where \( f(\cdot) \) is the implied (risk-neutral) probability density function for the random variable \( S_T \), the value of the underlying at the expiration date, we can use the volatility curve \( \sigma(\cdot) \) to calculate parameters about its distribution.

From arbitrage theory, we know that

\[
E[S_T] = \int_0^\infty K f(K) \, dk = F
\]

The variance is

\[
\text{var}[S_T] = E[S_T^2] - F^2
\]

and the entropy is

\[
H[S_T] = -E[\log f(S_T)]
\]

1. What is the option-implied (risk-neutral) variance of AAPL for 9/21/2018? (4 points)
2. What is the implied (risk-neutral) entropy of \( S_T \)? (4 points)
3. What is the implied (risk-neutral) probability that \( S_T = 0 \)? (2 points)

**Solutions**

It should be apparent from the functional forms of the BSM equation and the implied variance fit that the strike price is more conveniently represented in terms of “log-moneyness”:

\[
u = \log(K/F)
\]
in terms of which
\[
c(K) = d F \tilde{c}(u) = d F \left( \Phi \left( -\frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) - e^u \Phi \left( -\frac{u}{\tilde{\sigma}(u)} - \frac{1}{2} \tilde{\sigma}(u) \right) \right)
\]
\[
\sigma(K) = \tilde{\sigma} \left( \log \left( \frac{K}{F} \right) \right)
\]
for \( K > 0 \).

Let’s take a look at the first derivative before moving on to the probability density. The first derivative
of the call price (wrt strike) is the integral of the density, the probability distribution function:
\[
\int^K f(k) \, dk = c'(K) \cdot dF \tilde{c}'(u) = e^{-u} \tilde{c}'(u)
\]
\[
= \Phi \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) + \Phi' \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) \tilde{\sigma}'(u) - 1
\]

This is still an indefinite integral; i.e. it includes a constant of integration. Since \( K \to 0 \) is equivalent to
\( u \to -\infty \), and since \( \tilde{\sigma}(u) \) is positive for all \( u \), the definite integral is
\[
\int^{Fe^u}_0 f(K) \, dK = \Phi \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) + \Phi' \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) \tilde{\sigma}'(u) - P \{ S_T = 0 \}
\]
where
\[
P \{ S_T = 0 \} = \lim_{u \to -\infty} \Phi \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) + \Phi' \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) \tilde{\sigma}'(u)
\]

**Default of underlying**

In the case of our \( \tilde{\sigma}(\cdot) \), this limit is zero. More generally this is not necessary. In fact, underlying prices\(^2\) can
go to zero in the real world—thus the event \( \{ S_T = 0 \} \) has a non-zero probability under any risk-neutral mea-
sure equivalent to the physical measure—in which case all put options expire in the money. The premiums
on deep-out-of-the-money puts can indicate the risk-neutral probability of this outcome.

Note that in order for \( P \{ K = 0 \} > 0 \),
\[
\lim_{u \to -\infty} \frac{u + \frac{1}{2} \tilde{\sigma}(u)^2}{\tilde{\sigma}(u)} = \lim_{u \to -\infty} \tilde{\sigma}(u) + \frac{1}{\tilde{\sigma}'(u)} \quad (\star)
\]
must exist (applying L’Hôpital’s rule in the second step).

Our implied standard deviation is a special case of the five-factor Gatheral form:
\[
\tilde{\sigma}(u) = \sqrt{c_1 + c_2 u + \sqrt{c_3 + c_4 u + c_5 u^2}}
\]

For large negative arguments (low strikes), \( \tilde{\sigma}(u) \approx \sqrt{-c_2 + \sqrt{c_5}} \sqrt{-u} \) (assuming it is not zero), and (\star) diverges. Therefore the Gatheral parameterization is not consistent with a defaultable underlying\(^3\).

\(^2\) perhaps not for indexes, but certainly for equity share securities

\(^3\) potential research opportunity
Mean and variance

Differentiating again, we get the result for the risk-neutral density:

\[
f(K) = \frac{d}{du} \left( e^{-u} \tilde{c}'(u) \right) \frac{du}{dK} = \Phi' \left( \frac{u}{\tilde{\sigma}(u)} + \frac{1}{2} \tilde{\sigma}(u) \right) \left( \frac{1}{\tilde{\sigma}(u)} - \frac{2u\tilde{\sigma}'(u)}{\tilde{\sigma}(u)^2} + \frac{u^2\tilde{\sigma}'(u)^2}{\tilde{\sigma}(u)^3} - \frac{\tilde{\sigma}(u)\tilde{\sigma}'(u)^2}{4} + \tilde{\sigma}''(u) \right) du \frac{du}{dK}
\]

We can evaluate the expected value using integration by parts:

\[
E[S_T] = 0 \cdot P\{S_T = 0\} + \int_0^\infty K f(K) dK
\]

\[
= F \int_{-\infty}^\infty e^u \frac{d}{du} \left( e^{-u} \tilde{c}'(u) \right) du = F \left( \tilde{c}'(u) \right)|_{-\infty}^\infty - \tilde{c}(u)|_{-\infty}^\infty
\]

The only one of these four limits that is non-zero is \( \tilde{c}(-\infty) = 1 \), which corresponds to the forward value of a zero-strike call. Therefore \( E[S_T] = F \) as required.

The variance of the underlying terminal value is

\[
\text{var} [S_T] = F^2 P\{S_T = 0\} + \int_0^\infty (K - F)^2 f(K) dK
\]

\[
= F^2 \left( P\{S_T = 0\} + \int_{-\infty}^\infty (e^u - 1)^2 \frac{d}{du} \left( e^{-u} \tilde{c}'(u) \right) \right) du
\]

We can again use integration by parts (twice), noting that \( \lim_{u \to -\infty} e^{-u} \tilde{c}'(u) = P\{S_T = 0\} - 1 \),

\[
\text{var} [S_T] = F^2 \left( P\{S_T = 0\} + (e^u + e^{-u} - 2) \tilde{c}'(u) \right)|_{-\infty}^\infty - \int_{-\infty}^\infty 2 (e^u - 1) \tilde{c}'(u) du
\]

\[
= F^2 \left( 1 - \int_{-\infty}^\infty 2 (e^u - 1) \tilde{c}'(u) du \right) = F^2 \left( 2 \int_{-\infty}^\infty e^u \tilde{c}'(u) du - 1 \right)
\]

We can evaluate this numerically, getting in this case 0.06928 \times F^2 or about 1798.28. Before we move on, note that our final result for the variance depends only indirectly on the implied standard deviation. This is an important result. Converting back to strike prices, we get

\[
\text{var} [S_T] = \frac{2}{d} \int_0^\infty c(K) dK - F^2
\]

with a little manipulation, and noting that \( c(K) - p(K) = dF - dK \) from put-call parity, we can write this as

\[
\text{var} [S_T] = \int_0^\infty \left( c(K)/d - (F - K)^+ \right) + \left( p(K)/d - (K - F)^+ \right) dK
\]

This can be implemented in practice by simply summing up all of the option forward time-values scaled by the strike interval, rather than going through the intermediary step of determining an implied volatility curve.\(^4\)

\(^4\)Note that \( d \) and \( F \) can be estimated from call and put prices by simply fitting a linear relationship between \( c(K) - p(K) \) and \( K \) from the parity requirement. No exogenous information is needed—not even the underlying price.
Information

The entropy calculation is strictly a numerical exercise. I got about 5.0326 nats.

There is an explicit result for the entropy of a log-normal random variable. The density of a log-normal with mean $F$ and logarithmic standard deviation $\sigma_{LN}$ is

$$f_{LN}(K) = \frac{\exp\left(-\frac{1}{2\sigma_{LN}^2} \left(\log(K/F) + \frac{1}{2}\sigma_{LN}^2\right)^2\right)}{\sqrt{2\pi\sigma_{LN}K}}$$

The has variance $F^2 \left(e^{\sigma_{LN}^2} - 1\right)$.

The negative logarithm of the density is

$$-\log f_{LN}(K) = \frac{(\log(K/F) + \frac{1}{2}\sigma_{LN}^2)^2}{2\sigma_{LN}^2} + \log \left(\sqrt{2\pi\sigma_{LN}F}\right) + \log(K/F)$$

so a natural change of variables to evaluate the entropy is

$$x = \log\left(\frac{K}{F}\right) + \frac{1}{2}\sigma_{LN}$$

$$dx = \frac{dK}{K\sigma_{LN}}$$

whereby

$$H_{LN} = \int_{-\infty}^{\infty} \left(\log \left(\sqrt{2\pi\sigma_{LN}F}\right) - \frac{1}{2}\sigma_{LN}^2 + \frac{1}{2}x^2 + \sigma_{LN}x\right) e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}dx}$$

$$= \log \left(\sqrt{2\pi e \sigma_{LN} F e^{-\frac{1}{2}\sigma_{LN}^2}}\right)$$

If we identify this with the numerical result, we find that the log-normal with the same variance as the underlying has a entropy of about 5.1159 nats. So this is an inferior fit.

Obviously the random variable $S_T$ cannot be normal, because the support has a lower bound (at zero). But this lower bound is quite remote so let’s set that consideration aside. The normal is interesting for begin the maximal entropy distribution for a random variable with a given variance. And since

$$H_N = \log \left(\sqrt{2\pi e \sigma_N}\right)$$

we get that that entropy is about 5.1653 nats.
Figure 2: Implied risk-neutral density of AAPL on September 21, 2018, as of September 12, 2017. A lognormal (dashed line) and normal (dotted line) with the same variance are shown for comparison.