Gauss’ Divergence Theorem

1. Let $\mathcal{R}$ be the region in $\mathbb{R}^3$ bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$. Let $S$ denote the boundary of $\mathcal{R}$. Let $\mathbf{F}(x, y, z) = (y, x, z^2)$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{A}$.

**Solution.** We want to solve this using Gauss’ Theorem, so we’ll compute the divergence of $\mathbf{F}$. We get $\text{div} \, \mathbf{F} = (0) + (0) + (2z) = 2z$. Thus we have

$$\iint_S \mathbf{F} \cdot d\mathbf{A} = \iint_{\mathcal{R}} 2z \, dV.$$ 

We’ll convert to cylindrical coordinates, which gives

$$\int_0^{2\pi} \int_0^1 \int_0^{1} (2z) \, r \, dz \, dr \, d\theta.$$

The rest of this problem is simply to evaluate that integral (which turns out not to be so hard). We get

$$\int_0^{2\pi} \int_0^1 \int_0^{1} (2z) \, r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 r \left[ z^2 \right]_{z=0}^{z=1} \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (r - r^5) \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{1}{2} r^2 - \frac{1}{6} r^6 \right]_0^1 \, d\theta = 2\pi \left( \frac{1}{2} - \frac{1}{6} \right) = \frac{2\pi}{3}.$$ 

2. Let $\mathcal{R}$ be the region in $\mathbb{R}^3$ described by $x^2 + y^2 + z^2 \leq 1$. Use the divergence theorem to evaluate $\iiint_{\mathcal{R}} z^2 \, dV$.

**Solution.** Let $S$ be the unit ball of radius 1. We want to find some vector field $\mathbf{F}$ whose divergence is $z^2$. We’ll have to make a choice here, and we’ll decide to use the vector field $\mathbf{F} = (0, 0, \frac{1}{3} z^2)$. This makes our process easier because we now only need to integrate

$$\iint_S \left( 0, 0, \frac{1}{3} z^2 \right) \cdot d\mathbf{A}.$$ 

We can parameterize the unit ball in the usual way and do the integration. We get $\frac{4\pi}{15}$ for our final answer.

**Remark.** These questions are taken from Shannon’s worksheet from Spring 2014. You can see the original question (and answer) [here](#).