Determinants

1. Show that
\[
\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix}.
\]
What is their common value?

Solution.
\[
\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 6 \end{vmatrix} = -2
\]

2. Compute the determinant of \[
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{pmatrix}.
\]

Hint: For those of you who have seen some Linear Algebra, look carefully at the rows of this matrix!

Solution. You may use the basketweave method to evaluate the determinant of the matrix. Alternatively, you may also notice that the third row is the sum of the first two, implying that the determinant is 0.

3. Given two vectors \((a, b)\) and \((c, d)\), how can we express \((a, b) \cdot (c, d)\) as a determinant? (That is, write down the matrix whose determinant is equal to \((a, b) \cdot (c, d)\).)

Solution. There are several ways to do this, but one way to accomplish this is
\[
\begin{vmatrix} a & -b \\ d & c \end{vmatrix} = ac - (-b)(d) = ac + bd = (a, b) \cdot (b, d)
\]
Cross Products

4. Compute \((j - 2k) \times (i + k)\).

Compare this to the value that I got for \((i + k) \times (j - 2k)\).

Solution.

\[
(j - 2k) \times (i + k) = \begin{vmatrix} i & j & k \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{vmatrix}.
\]

Students of linear algebra may note that this is similar to the matrix that I got, except that the rows are swapped. Since swapping rows negates the determinant, we know that the determinant of this matrix will be negative what we got for the original matrix.

If you don’t notice this, however, you can compute the determinant using expansion by minors or the basketweave method. We get

\[i - 2j - k = i - 2j - k.\]

If we compare the vectors we found for \((j - 2k) \times (i + k)\) and \((i + k) \times (j - 2k)\), we notice that one is negative the other. This is because cross product (unlike usual multiplication) is not commutative.

5. Compute \(\|(j - 2k) \times (i + k)\|\).

Solution. Since we already know that \((j - 2k) \times (i + k) = i - 2j - k\), we need only compute the length of that vector. We get

\[\|i - 2j - k\| = \sqrt{1^2 + (-2)^2 + (-1)^2} = \sqrt{1 + 1 + 2} = \sqrt{4} = 2.\]

On Friday, you will learn another way to compute the length of a cross product.

6. What is \(i \times i\)? \(i \times j\)? (Can you guess what \(i \times k\) will be?)

Solution. These boil down to computing more determinants. You should get

\[i \times i = 0\]
\[i \times j = k\]
\[i \times k = -j\]