Conservative Vector Fields

**Theorem 1.** The vector field $F = \nabla f$ is a conservative vector field if and only if for any two paths $c_1, c_2$ with the same endpoints,

$\int_{c_1} F \cdot ds = \int_{c_2} F \cdot ds$

**Theorem 2.** Let $F$ be a conservative vector field with potential function $f$. Let $c$ be a path with endpoints $a$ and $b$. Then

$\int_c F \cdot dS = \int_c F \cdot ds$

**Exercises.**

1. Let $F(x, y, z) = \left( \frac{2x^2 y + 4y^2}{4x^3 + 2y + 8xy} \right)$. Evaluate $\int_c F \cdot dS$ where $c(t) = \left( \frac{\cos \pi t}{t^3 + 1} \right)$

2. Is $F(x, y, z) = \left( \frac{2xyz^3 + z}{x^2z^3 - e^z \sin y} \right)$ a conservative vector field? Why or why not?

(You can probably figure this out by inspection, but I want you give some mathematical evidence, e.g. if the answer is “no,” then do some line integrals.)

3. Suppose $F$ is any conservative vector field and $C$ is any simple, closed curve. Evaluate $\int_C F \cdot dS$. 
In-Class Examples

1. Let’s integrate
\[ \int_{\frac{1}{2}}^{\frac{5}{2}} 2t \sin \pi t + \pi t^2 + 4\pi (\sin \pi t)(\cos \pi t) \, dt. \]

Our theorem tells us that if we can rephrase this as a path integral AND the vector field is conservative, then we can choose an easier path and get an easier integral. In this case we have \( F = \left( \frac{2xy}{x^2 + 4y} \right) \) and \( c(t) = (t, \sin \pi t) \) for \( t \in \left[ \frac{1}{2}, \frac{5}{2} \right] \). Our v.f. is conservative with potential function
\[ f(x, y) = x^2 y + 2y^2. \]

So our theorem applies! Endpoints of \( c(t) \)?
\[ c \left( \frac{1}{2} \right) = \left( \frac{1}{2}, 1 \right) \]
\[ c \left( \frac{5}{2} \right) = \left( \frac{5}{2}, 1 \right). \]

So instead let’s use the straight line path \( d(t) = (t + \frac{1}{2}, 1) \) for \( t \in [0, 2] \). So our integral becomes
\[ \int_{0}^{2} 2t + 1 \, dt = t^2 + t \bigg|_{0}^{2} = 6. \]

But we can say something even faster. Theorem 2 tells us that we only need to look at the potential function and the end points...
\[ f \left( \frac{5}{2}, 1 \right) - f \left( \frac{1}{2}, 1 \right) = \frac{33}{4} - \frac{9}{4} = 6. \]

2. By our theorem, \( F(x, y) = (-y, x) \) is not a gradient vector field. Consider
\[ c_1(t) = (t, t) \quad \text{for } t \in [0, 1] \]
\[ c_2(t) = (t, t^2) \quad \text{for } t \in [0, 1]. \]

Doing the integrals: we get 0 and \( \frac{1}{4} \). So by Theorem 1, this vector field is not conservative.