Surfaces

Definition. We say that a parameterization \( \phi(s,t) \) of a surface \( S = \phi(D) \) is smooth at a point \((s_0, t_0) \in D\) if the map \( \phi \) is \( C^1 \) near \((s_0, t_0) \) and the normal vector is nonzero. That is

\[
N(s_0, t_0) = T_s(s_0, t_0) \times T_t(s_0, t_0) \neq 0.
\]

If \( \phi(s, t) \) is smooth at every point \( \phi(s_0, t_0) \in S \), then we say that \( S \) is a smooth parametrized surface.

Example. Consider the surface defined by \( z = f(x, y) = x^2 + 2xy + y^3 \). Give a parameterization of \( f \).

\[
\phi(s, t) = \begin{pmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{pmatrix} = \begin{pmatrix} \_ \\ \_ \\ \_ \end{pmatrix}.
\]

Compute \( N(s, t) \) at \((s, t) = (2, 3)\).

\[
N(2, 3) = \frac{\partial \phi}{\partial s}(2, 3) \times \frac{\partial \phi}{\partial t}(2, 3) = \det \begin{pmatrix} i & j & j \\ 1 & 0 & \_ \\ 0 & 1 & \_ \end{pmatrix} = \]

Is \( \phi(s, t) \) smooth at \((s, t) = (2, 3)\)? More generally, is \( S = \phi(s, t) \) a smooth parametrized surface? Why or why not?

Bonus. What is the surface area of this surface over the square bounded by \( 0 \leq s \leq 2, \ 0 \leq t \leq 3 \)?

Your Turn! Let \( \phi(s, t) = (e^s \cos t, e^s \sin t, t) \) for \( s \in [0, 1] \) and \( t \in [0, \pi] \).

1. Find \( T_s \times T_t \).

2. Find the (equation for) the tangent plane to \( S \) when \( (s, t) = (0, \pi) \).

3. Find the area \( \phi(D) \) where \( D = [0, 1] \times [0, \pi] \).
Vicki’s Thanksgiving Feast

Vicki’s is making a Carrion Cake. Her cake pan can be parametrized by
\[ \phi(s,t) = \begin{pmatrix} (5 + 2 \cos t) \cos s \\ (5 + 2 \cos t) \sin s \\ 2 \sin s \end{pmatrix} \]
for \(0 \leq s \leq 2\pi\) and \(\pi \leq t \leq 2\pi\).

1. What is the tangent plane to her pan at \((x,y,z) = \begin{pmatrix} 3 \sqrt{3} \\ 3 \\ 1 \end{pmatrix}\)?

2. Given that she doesn’t ice the bottom of her cake, how much icing does Vicki need to completely cover her cake? (When she takes the cake out of the pan, the surface of the cake is almost identical to the description of the pan, but now \(0 \leq s \leq 2\pi\) and \(0 \leq t \leq \pi\).)