Notes from Humphreys, Reflection Groups and Coxeter Groups,

Section 2.3

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This is a writeup discussing section 2.3 of Humphreys, Reflection Groups and Coxeter Groups. I include additional material outside the textbook to elaborate on some of the explanations.

1 Introduction

Working towards a classification of finite Coxeter groups, our approach is to abstract some of the features that arise naturally when studying such a group. We’ve started to do this already – in Chapter 1 we defined an abstract root system using properties (R1) and (R2) from section 1.2. In this section we define another combinatorial object:

Definition. An (abstract) Coxeter graph $\Gamma$ is a finite simple undirected graph whose edges are each labeled with a positive integer greater than or equal to 3, or $\infty$.

We saw that every finite Coxeter system $(W,S)$ determines a Coxeter graph. To ask whether one can go in the other direction, we will describe several pieces of data that are equivalent to the Coxeter graph of $(W,S)$:

1. a symmetric matrix $A = (a(s,s'))_{s,s'\in S}$ giving the inner products of simple roots in a fixed simple system $\Delta$ (the inner product taken with respect to an orthonormal basis of $V$);

2. a symmetric, bilinear form giving the inner product of two vectors written in the basis $\Delta$;

3. a quadratic form giving the ‘norm’ of a vector written in the basis $\Delta$.

We now show that these data are equivalent. Fix $(W,S)$ and let $\Gamma$ be the Coxeter graph of $W$ as defined in section 2.1. Suppose further that the roots are unit vectors. Then let $A = (a(s,s'))_{s,s'\in S}$ be the matrix defined by $a(s,s') = -\cos(\frac{\pi}{m(s,s')})$. We have that $a(s,s')$ is symmetric since $m(s,s')$ is.

We claim that $a(s,s') = \langle \alpha_s, \alpha_{s'} \rangle$. It suffices to show these values are equal in the root system $\Phi_{s,s'}$ of the parabolic subgroup $W_{s,s'}$. Here, $ss'$ has order $m(s,s')$, so it is rotation by the angle $\frac{2\pi}{m(s,s')}$. We also know that the angle between $\alpha_s$ and $\alpha_{s'}$ is an obtuse or right angle. These conditions imply that $\alpha_s$ and $\alpha_{s'}$ have angle $\pi - \frac{\pi}{m(s,s')}$ between them. Thus the dot product of these (unit!) vectors is $|\alpha_s||\alpha_{s'}|\cos(\pi - \frac{\pi}{m(s,s')}) = -\cos(\frac{\pi}{m(s,s')})$, as required.
Now given $A$, we define a bilinear form as follows: given two row vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, define $\langle x, y \rangle := xAy^T$. (This is written differently in the textbook as there, $x$ and $y$ are column vectors. If the simple roots are ordered $\alpha_1, \ldots, \alpha_n$, then $\langle x, y \rangle$ is the inner product of the vectors $\sum_{i=1}^n x_i\alpha_i$ and $\sum_{i=1}^n y_i\alpha_i$, taken with respect to some orthonormal basis.

Then given a symmetric bilinear product, $\langle \cdot, \cdot \rangle$, we get a quadratic form $x \rightarrow \langle x, x \rangle$.

We can go in the other direction as well: starting with any quadratic form $f$ we can recover a symmetric matrix $A$ so that $f(x) = xAx^T$. Then the bilinear form is $\langle x, y \rangle = xAy^T$, and the $m(s, s')$ can be recovered as well if the entries have absolute value at most one. Of course, starting with an arbitrary symmetric matrix $A$, the $m(s, s')$ will not always be positive integers, but for our considerations that won’t be an issue. We’ll be looking at symmetric matrices of Coxeter graphs and trying to determine which ones come from finite Coxeter groups.

Here’s a good litmus test. If $\Gamma$ comes from a finite Coxeter group acting on a finite root system, then the quadratic form is the norm function on vectors written in the basis $\Delta$. That function ought to be positive on nonzero vectors. If it is, the symmetric matrix $A$ is called positive definite. If it isn’t, we have a problem. Though it will turn out to be useful to look at Coxeter graphs whose quadratic forms are nonnegative on nonzero vectors, in which case we say $A$ is positive semidefinite.

It is well-known that a matrix $A$ is positive definite (positive semi-definite) if and only if the principal minors are positive (nonnegative). The principal minors are the determinants of the square submatrices of $A$ using rows $1$ through $j$ and columns $1$ through $j$, for $1 \leq j \leq n$. 