Section 1.5: A Characterization

1. Theorem for Identifying Coexter Systems

Standing Assumption. Let $W$ be any group generated by a (generating) set $S$ of elements such that $s^2 = e$ for all $s \in S$.

This imposes a definition of reduced expressions and length (as in previous sections). However, this is more general than the assumptions from the previous section so it may be that not all the properties of the length function will hold.

Question. How can we identify if $(W, S)$ is a Coxeter system?

First we revisit the notion of “exchange property” from before. Previously we gave a theorem that said all Coxeter groups satisfied an exchange property. Now we simply state it as something that could be satisfied (we will reference it in the main theorem from this section).

Let $(W, S)$ be described as above, Let $w = s_1 s_2 \ldots s_k$ be a reduced expression. Let $s \in S$. If $\ell(sw) < \ell(w)$ then the exchange property says that there exists $i \in [k]$ such that

$$sw = s_1 s_2 \ldots \hat{s}_i \ldots s_k$$

where the hat denotes omission. As we will see in the following theorem, a pair $(W, S)$ that satisfies the exchange property is a Coxeter group.

Theorem 1. Let $(W, S)$ be as above. Then TFAE.

(a) $(W, S)$ is a Coxeter System.
(b) $(W, S)$ has the exchange property.
(c) $(W, S)$ has the deletion property (see Section 1.4).

2. Extended Example: $S_n$

Standing Assumption. Let $S$ denote our favorite generating set for $S_n$

$$S = \{s_i = (i, i + 1) \mid i \in [n-1]\}.$$ 

Let $\ell_A(\cdot)$ denote a length function on $S_n$ with respect to $S$.
**Definition** (Inversion). Let $\sigma \in S_n$. Then $\text{inv}(\sigma) = \#\{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}$.

**Example 1.** Let $\sigma = 231 \in S_3$ (written in one-line notation). The descents are $(1, 3)$ and $(2, 3)$. So $\text{inv}(\sigma) = 2$.

**Proposition.** Let $\sigma \in S_n$. Then $\ell_A(\sigma) = \text{inv}(\sigma)$.

As an aside (related to Esther’s question), the descent set of a permutation $\sigma$ is a position $i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$.

**Example 2.** Let $\sigma = 123 \in S_3$. The only descent is 2, since $\sigma(2) = 3$ and $\sigma(3) = 1$ so $\sigma(2) > \sigma(2+1)$. Note that 1 is not a descent since $\sigma(1) = 2$ and $\sigma(2) = 3$. That is, $\sigma(1) < \sigma(1+1)$.

We can think of descents as inversions of the form $(i, i+1)$. So in a way there is a connection between descents and inversions. However, the number of descents will be less than or equal to the number of inversions.

**Proposition.** Let $\sigma \in S_n$. Then $D_R = \{\sigma \in S \mid \sigma(i) > \sigma(i+1)\}$.

**Proposition.** The pairing $(S_n, S)$ is a Coxeter system of type $A_{n-1}$. 