Chapter 2: The Bruhat Order

Section 2.4: Parabolic Subgroups and Quotients

The focus of this section is to understand a special collection of subgroups of a Coxeter group. These are the subgroups that are also Coxeter groups.

**Example 1.** The symmetric group \( S_n \) is generated by the transpositions \((i \ i + 1)\) for \( i \in [n - 1] \) so any subgroup of \( S_n \) generated by a subset of those transpositions will again be a Coxeter system. In particular, take \( n \geq 3 \). Then consider the subgroup generated by \((1 \ 2)\) and \((2 \ 3)\). This is again a Coxeter group (indeed, it is isomorphic to \( S_3 \)).

**Non-example.** The alternating group on \( n \) elements is generated by odd permutations (i.e. permutations that can be written by an odd number of permutations). In particular, we have

\[
\text{(alternating group on } n \text{ elements)} = \langle (123), (124), \ldots, (12n) \rangle.
\]

This forms a subgroup of \( S_n \) but it is not generated by a collection of order 2 elements. Thus the alternating group is not a Coxeter group.

**Standing Assumption.** Throughout this section, we’ll assume that \((W, S)\) is a (possibly infinite) Coxeter system. In the special case where \( S \) is finite, we will put some ordering on \( S \) to get \( S = \{s_1, \ldots, s_n\} \).

**Definition** (Parabolic Subgroup). Let \( J \subset S \) and let \( W_J \) denote the reflection group generated by \( J \). We say that \( W_J \) is a parabolic subgroup.

**Remark** (Naming Conventions). Humphreys has a post on Math Stack Exchange where he says that parabolic subgroups are a thing in Lie theory and that if we all understood Lie theory, then the naming convention feels natural\(^1\). According to Eric, the Borel subgroup here is the trivial group. If you have more questions, I refer you to him.

At this point, Björner and Brenti get very excited and start appending \( J \)’s onto everything (and in every imaginable position: by the end of the section we’ll have \( W_J, W^J \) and \( JW \) which all mean different things). For the moment, I’ll just say that we can re-define a

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\(^1\)See: https://mathoverflow.net/questions/168033/coxeter-groups-parabolic-subgroups
length function for $W_J$ by writing things in terms of the generators associated to elements of $J$. We’ll call it $\ell_J$.

There are some obvious consequences of this definition that I’ve set you up to believe (possibly without proof). For example: $(W_J, J)$ was a gain a Coxeter system. If that wasn’t true, then we’d want to re-write the definition until it was. That was the “point.”

Below are some other things that are true about parabolic subgroups (mostly from Proposition 2.4.1 of Björner and Brenti).

1. For $w \in W_J$, the length function inside $\ell_J(w) = \ell(w)$.
2. $W_I = W_J$ if and only if $I = J$.
3. $W_I \cap W_J = W_{I \cap J}$
4. $\langle W_I \cup W_J \rangle = W_{I \cup J}$
5. The Coxeter diagram of $W_J$ can be obtained from the Coxeter diagram of $W$ by deleting the nodes (and edges) associated to $S - J$.

There are some more fiddly properties of $W_J$. First, there are minimal coset representatives of the $wW_J$ cosets in $W$ (Coro 2.4.5). Additionally, in the case were $W_J$ is finite ($W$ itself need not be finite), each coset has a maximal representative (Coro 2.4.5) and $W_J$ has a maximal (in Bruhat order) element $w_0$. This maximal element (what Humphreys called the “longest word”) will be important in the following discussion in which we use the subgroup structure of $W_J$ to factor things in $W$.

**Factoring Words using $W_J$.**

The goal of this section is to produce ways to factor words $w \in W$ into a word in $W_J$ and a word in some other group (which we will specialize). We will also see how this implies a decomposition of $W$ via factorizations. First we need some more notation.

**Definition** (Descent Class of $W$). Let $I, J \subset S$. Then a (right) descent class of $W$ is

$$D^R_J = \{ w \in W \mid I \subseteq D_R(w) \subseteq J \}.$$ 

A left descent class can be defined by changing the $D_R(w)$ to be $D_L(w)$.

**Remark.** In the special case where $I = J$ we can simply write $D_I$. Just *be careful* and don’t name your subset $R$ or $L$. Trust me.

**Definition** (Quotient). A quotient $W^J$ of $W$ is given by

$$W^J = \{ w \in W \mid ws > w \text{ for all } s \in J \}$$

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2Björner and Brenti only prove this in the forward direction but I am reasonably convinced that the backwards direction is also true. Counterexamples welcome. :P
This is equivalent to saying that $W^J = D^{S-J}_\emptyset$ since
\[
D^{S-J}_\emptyset = \{ w \in W \mid \emptyset \subseteq D_R(w) \subseteq (S - J) \} = \{ w \in W \mid D_R(w) \subseteq (S - J) \} = \{ w \in W \mid \text{every } s \in S \text{ such that } \ell(ws) < \ell(w) \text{ is contained in } (S - J) \} = \{ w \in W \mid ws > w \text{ for all } s \in J \}.
\]
We should note that it is never true that $ws = w$ so even when we flip the inequality, it remains strict.

**Remark.** The quotient is a collection of minimal length left cosets $wW^J$ and we will take advantage of that when we look at factorizations of words in $W$.

**Definition** (Another definition of quotient). There is an inverse version (or left-handed version) of $W^J$, which we denote by either $(W^J)^{-1}$ (or $JW$). This is defined to using the left descent sets. We say
\[
JW = \{ w \in W \mid D_L(w) \subseteq (S - J) \}.
\]
This gives minimal length representatives of right cosets. As you might have suspected, this will give us a left-handed version of the factorization we got using the quotient.

**Theorem 1** (Nice Factorizations via Parabolic Subgroups). Let $J \subseteq S$. Then for all $w \in W$, there exists a factorization of $w = w^Jw_J$ where $w^J \in W^J$ and $w_J \in W_J$. Furthermore this factorization does not reduce. That is $\ell(w) = \ell(w^J) + \ell(w_J)$.

**Example 2.** Let $W = S_3$. Then $S = \{(1 2), (2 3)\}$. Take $J = \{(1 2)\} \subset S$. Then
\[
W_J = \{ ( ) \}, (1 2) \}
W^J = \{ (2 3), (1 2)(2 3) \}.
\]
Let’s use this to factor $w = (2 3)(1 2)$. You’ll notice that $w$ is in neither $W_J$ nor $W^J$. However, if we take $w_J = (1 2)$ and $w^J = (2 3)$ then we get
\[
w = (2 3)(1 2) = w^Jw_J.
\]

**Theorem 2** (Left-handed Version of the Previous Theorem). Let $J \subseteq S$. Then for all $w \in W$, there exists a factorization of $w = w^Jw_J$ where $w_J \in W_J$ and $w^J \in ^JW$. Furthermore this factorization does not reduce. That is $\ell(w) = \ell(w_J) + \ell(^Jw)$.

**Example 3.** As in the previous example, let $W = S_3$. Then $S = \{(1 2), (2 3)\}$. Again take $J = \{(1 2)\} \subset S$. Then
\[
W_J = \{ ( ), (1 2) \}
\]
\[
^JW = \{ (2 3), (2 3)(1 2) \}.
\]
Let’s use this to factor $w = (2 3)(1 2)$ (as before). You’ll notice that $w$ is not in $W_J$ but it is in $^JW$. So take $^Jw = (2 3)(1 2)$ and $w_J = ( )$ then we get
\[
w = ( )(2 3)(1 2) = w_J^Jw.$
We take a leap now, and move from viewing $W^J$ as a subset of $W$ and view the $[blah]^J$ as an operation on an arbitrary set. We sort of saw this when we looked at the left-handed version of $W^J$ but this time we’ll take two different subsets $J$ and $J'$ of the generating set and look at $W^J_J'$. 

**Example 4.** Let’s look at $W = S_4$. We’ll take $J = \{(1 2), (2 3)\}$. Then

$$W_J = \{( ), (1 2), (2 3), (1 2)(2 3), (2 3)(1 2), (1 2)(2 3)(1 2)\} \cong S_3$$

Now take $J' = \{(1 2)\}$. Let’s compute $(W_J)^{J'}$. We get

$$(W_J)^{J'} = \{w \in W_J | ws > w \text{ for all } s \in J'\} = \{( ), (2 3), (1 2)(2 3)\}.$$ 

We can use this construction to produce a decomposition-like bijection for a finitely-generated Coxeter group. Let us consider a finite generating set $S$ for a Coxeter system. We can index the generators by $[n]$ (see the standing assumptions up at the top). For each $i \in [n]$, we can define $Q_i = (W_{\{s_1,\ldots,s_i\}})^{\{s_1,\ldots,s_{i-1}\}}$. In particular $Q_1 = W_{\{s_1\}}^e = \{e, s_1\}$ and for all $i \geq 2$, these $Q_i$ are the system of minimal left coset representatives of $W_{\{s_1,\ldots,s_{i-1}\}}$ inside $W_{\{s_1,\ldots,s_i\}}$.

**Example 5.** Let $W = S_4$. Then $Q_1 = \{( ), (1 2)\}$. In the previous example, we computed $Q_2$. We got

$$Q_2 = \{( ), (2 3), (1 2)(2 3)\}.$$ 

Using these $Q_i$-thingies we can construct a decomposition-type map. Simply apply the factorization theorems repeatedly. We obtain the following corollary.

**Corollary 1.** The product map $f : Q_1 \times Q_2 \times \cdots \times Q_n \to W$ defined by $f(q_1, \ldots, q_n) = q_n \cdots q_1$ is a bijection satisfying a non-reduction property. Explicitly

$$\ell(q_nq_{n-1}\cdots q_1) = \ell(q_1) + \ell(q_2) + \cdots + \ell(q_n).$$ 

**Example 6.** We return to $W = S_4$. We have already seen that

$$Q_1 = \{( ), (1 2)\}$$

$$Q_2 = \{( ), (2 3), (1 2)(2 3)\}.$$ 

We have not yet computed $Q_3$, but for the purposes of this example, we will simply say that it is the final $Q_i$ and that is is the set of minimal-length coset representatives of $S_3$ inside $S_4$. The corollary tells us that there is a bijection $f : Q_1 \times Q_2 \times Q_3 \to W$ defined by $f(q_1, q_2, q_3) = q_3q_2q_1$. That is: every $w \in W$ can be decomposed into $w = q_3q_2q_1$ where $q_i \in Q_i$. 

4
**Extended Example: \( S_n \).**

**Notation Update.** I’m changing the way that I denote permutations. Starting now permutations will be written in one-line notation (instead of cycle notation).

In the context of the symmetric group, parabolic subgroups are also called Young subgroups. With this in mind, we’ll specialize the definitions of \( W_J \) and \( W^J \) (from the previous section) as well. We will focus on subsets of the generators of the form \( J = S - \{ s_k \} \), where \( s_k \) denotes the transposition that swaps \( k \) and \( k+1 \).

**Example 7** (Björner and Brenti). Let \( n = 6 \) and take \( J = \{ s_2, s_3, s_5 \} \). Then

\[
(S_n)^J = \{ x \in S_6 \mid x_2 < x_3 < x_4 \text{ and } x_5 < x_6 \}
\]

**Lemma 1** (2.4.7). Fix \( k \in [n] \). Suppose \( J = S - \{ s_k \} \). Then

\[
(S_n)_J = \text{Stab}([k]) \cong S_k \times S_{n-k} \tag{1}
\]

\[
(S_n)^J = \{ \sigma \in S_n \mid \sigma_1 < \cdots < \sigma_k \text{ and } \sigma_{k+1} < \cdots < \sigma_n \}. \tag{2}
\]

**Proof.** (1) The parabolic subgroup \((S_n)_J\) is precisely the set of elements that do not contain \( s_k \) in any (reduced) factorization. That is, we are able to obtain any permutation by permuting the first \( k \) entries. Similarly we can permute the last \( n-k \) entries in any way we want. However, since we are unable to swap \( k \) and \( k+1 \), we cannot obtain permutations where entries in the first \( k \) positions are swapped with entries in the last \( n-k \) positions.

(2) Following the definition from the factorization section of this write-up, we have

\[
(S_n)^J = \{ \sigma \in S_n \mid \sigma s > \sigma \text{ for all } s \in J \}.
\]

Since \( J = S - \{ s_k \} \), we have

\[
(S_n)^J = \{ \sigma \in S_n \mid \sigma s > \sigma \text{ for all } s \in S \text{ except } s_k \}
= \{ \sigma \in S_n \mid \text{the only } s \in S \text{ such that } \sigma s < \sigma \text{ is } s_k \}
= \{ \sigma \in S_n \mid \sigma_1 < \cdots < \sigma_k \text{ and } \sigma_{k+1} < \cdots < \sigma_n \}.
\]

When \( J = S - \{ s_k \} \), we will denote \((S_n)^J\) by \( S_n^{(k)} \). That is

\[
S_n^{(k)} = \{ \sigma \in S_n \mid \sigma_1 < \cdots < \sigma_k \text{ and } \sigma_{k+1} < \cdots < \sigma_n \}.
\]
Remark. The reader may notice that $\sigma \in S_n^{(k)}$ is uniquely determined by the first $k$ entries (and indeed we need not specify the order of those entries, since they are always ordered in increasing order). We will use this observation in understanding the Bruhat order restricted to $S_n^{(k)}$. This implies that there is a one-to-one correspondence between subsets of $[n]$ and the elements of $S_n^{(k)}$.

We can construct a map $f : S_n \to S_n^{(k)}$ by rearranging the first $k$ entries and the last $n - k$ entries of $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$ so that they are (each separately) in increasing order. Let $x^J$ be such a rearranged version of $x$. Then we say that $x \mapsto x^J$ under $f$.

Example 8. Let $n = 7$ and let $k = 3$. Then $J = \{1, 2, 4, 5, 6\}$. Let us compute $f(7125346)$. We have to arrange $\{7, 1, 2\}$ and $\{5, 3, 4, 6\}$ in increasing order. Thus $x^J = 1273456$.

We can extend this definition in two ways: we can use a more general $J$ than $S - \{s_k\}$ and we can define a dual-type construction for $w^J$, which we denote $J w$. The following two examples illustrate these two extensions.

Example 9 (More General $J$). Let $n = 7$ and let $J = \{1, 2, 4, 6\}$. The “missing” generators are 3 and 5. Let us compute $f(7125346)$. We have to arrange $\{7, 1, 2\}$, $\{5, 3\}$, and $\{4, 6\}$ in increasing order. Thus $x^J = 1273546$.

Example 10 (Right-sided version). Let $n = 7$ and let $J = \{1, 2, 4, 6\}$. Let us compute $J w$ for $x = 7125346$. In this case we have to arrange $\{7, 1, 2\}$, $\{5, 3\}$, and $\{4, 6\}$ in decreasing order. We attain $J w = 1273546$.

We now turn to the Bruhat order restricted to $S_n^{(k)}$. As it turns out, when we restrict to this particular quotient, the Bruhat order is easy to describe. We give a characterization in the following proposition.

Proposition (2.4.8). Let $x, y \in S_n^{(k)}$. The following are equivalent.

1. $x \leq y$
2. For all $i \in [k]$, we have $x_i \leq y_i$,
3. For all $ik + 1 \leq i \leq n$, we have $x_i \leq y_i$.

We can apply this theorem to the remark after the definition of $S_n^{(k)}$. We identify $S_n^{(k)}$ (under the Bruhat order) with the family of $k$-subsets of $[n]$ under the product order on $k$-tuples. Sometimes we denote the family of $k$-subsets of $[n]$ under the product order on $k$-tuples by $L(k, n - k)$. The poset $L(k, n - k)$ is isomorphic to Young’s lattice (only the Young diagrams that fit into a $k \times (n - k)$ box).