Section 2.5 is devoted to the somewhat surprising fact that parabolic quotients $W^J$, that is, the set of minimal coset representatives for a parabolic subgroup $W_J$, actually have a natural poset structure induced from the Bruhat order on the whole group $W$.

The main thrust of this section is to convince you that the poset $W^J$ on the quotient enjoys many of the nice features of the Bruhat order itself: namely that it is directed, graded by length, and satisfies a Chain Property (see Section 2.2). The proofs, in all cases, are rather straightforward.

The only result that falls outside this program is Proposition 2.5.4:

**Proposition (2.5.4).** Let $(W, S)$ be finite and $J \subseteq S$. Then $\alpha : W \to W$ defined by
\[
\alpha(x) = w_0 x w_0^J,
\]
where $w_0^J$ is the longest element of $W_J$, restricts to an antiautomorphism $W^J \to W^J$.

The authors call this “a remarkable combinatorial symmetry”. Its significance is not particularly clear yet, but I agree with the description. A tiny remark: considering its superficial similarity to the conjugation map, it is perhaps a bit surprising that it ends up being an antiautomorphism.

On the next page we compute an example of the quotient poset.

**Missteps to avoid (which I did not):**

- Proposition 2.5.2 (below for reference) does not imply that the Hasse digram of $W^J$ is a connected subgraph of the Hasse diagram of $W$ (i.e. that $u \prec w$ in $W^J$ implies $u \prec w$ in $W$. This fact is instead implied by the Chain Property for quotients.

  **Proposition (2.5.2).** Suppose $u \in W^J$, $w \in W$, and $u \prec w$ (in $W$). Then either $w = us$ for some $s \in J$, or $w \in W^J$.

- In Proposition 2.5.4 (on the first page) the conclusion is that $\alpha$ is an antiautomorphism. This is meant in the sense of posets; there is generally no group structure on $W^J$.

- In the example on the next page, I will make an extended remark about my troubles with handedness.
**Example.** Let $W = S_4$ and $J = \{s_1, s_3\}$. Then the following diagram shows $W^J$ as a subposet of $W$:

The most pressing observation is that this diagram is *not consistent* with Lemma 2.4.3 or Proposition 2.4.8. The reason for this is related to handedness issues. In particular, although the labels written here are *correct* (i.e., internally consistent with the order relations), they are not consistent with the ones in the diagram on page 31.

If you wish to adjust the diagram to fully align with the Björner-Brenti conventions, the easiest way is to *preserve the shape of $W^J$* while changing the labels on the entire $W$ by reversing the order of the generators at each point.

The exact issue is the following: the diagram above uses the Functional for multiplication in the Coxeter group and I assumed that they would be using left actions, because seriously who uses right actions? The answer is that Björner and Brenti do: since they also use Functional Convention, but to agree with the one-line notation they also use the Replacement Convention — the result is, indeed, a right action.

(After spending far more time banging my head on this issue than I care to admit, I actually like their choice; it seems to me that Replacement is optimal for sane cycle/one-line conversions. If you *really* want a left action I would recommend giving up Functional instead.)
A few remarks on how this diagram came into the world:

Perhaps it is too obvious to state, but I did not use either Lemma 2.4.3 or Proposition 2.4.8 to draw this diagram, which would have been the smart thing to do, simply because I did not notice them. Instead I used the proof of Proposition 2.4.4, which gives an algorithmic, if rather inefficient, way of determining the elements of $W^J$ in general.

The reduced expressions chosen for this diagram can be treated as essentially random: there is no reason at all to prefer the ones I’ve written here over any others. However, the process that I was using led me to the following empirical observation:

**Fact.** For all $w \in W$ there exists $u \in W$ and $s \in S$ such that $u \triangleleft w$ and $us = w$.

This is not completely trivial; working immediately from the definition of the Bruhat order we only obtain the weak version which permits $s \in T$. But upon further investigation, this Fact is essentially the Chain Property when the lengths $u$ and $w$ differ by 1 (together with its consequence that the Bruhat order is graded by length).

Further remarks on handedness, which are amusing, if not terribly relevant:

If you use the Reading Convention to write down the correct one-line notation labels for the diagram I gave above, you discover the modified version of Proposition 2.4.8 which holds in the presence of a right action: rather than requiring $x_i \leq y_i$ for $i \leq k$, for instance, you require that the numbers 1 through $k$ be written in increasing order; similarly $k + 1$ through $n$ must be written in increasing order.

The modified version of Lemma 2.4.3 is fairly predictable; rather than ending with $s \in J$, the obstruction is now starting with $s \in J$.

You may be tempted to assume that in the presence of a left action, the labels that I have marked on the diagram above are not the minimal coset representatives of $W_J$. This is morally true but technically false: they continue to be minimal coset representatives, but for the right cosets instead of the left cosets, or in other words, it is $W^J$ if you replace $D_R(w)$ in Definition 2.4.2 with $D_L(w)$. 