1 2.2

This section discusses basic properties of the Bruhat order. I will list most of these properties as well as examples of them. The examples will mostly all come from \( S_3 \) and \( S_4 \). A picture of the complete Bruhat order for \( S_4 \) is on page 31 of our text.

In the first few properties, we get a sense for

**Lemma 1.** Let \( u, w \in W \) with \( u \neq w \), with one reduced expression of \( w \) being \( s_1 \cdots s_q \) such that there exists a reduced expression of \( u \) that is a subword of this expression of \( w, u = s_{i_1} \cdots s_{i_k} \) for \( 1 \leq i_1 < \cdots < i_k \leq 1 \). Then, there exists \( v \in W \) such that

- \( v > u \)
- \( \ell(v) = \ell(u) + 1 \)
- Some reduced expression for \( v \) is a subword of \( s_1 \cdots s_q \).

For example, for \( W = S_4 \) let \( w = s_3s_2s_3 \) (which is also \( s_2s_3s_2 \)) and let \( u = s_3 \). Then, using the language of the lemma, we can take \( v = s_2s_3 \) or \( s_3s_2 \). We have that \( v \) in fact covers \( u \) and \( w \) covers both choices of \( v \) which we can see in a small portion of the Bruhat order of \( S_4 \). If my choice of \( w \) had longer length, we would have more elements in the intervals between \( v \) and \( w \). Note that this Lemma did not yet guarantee that \( u < w \) (but of course, we will get there/ perhaps induction or iterative use of the Lemma will be enough?).

\[
\begin{array}{c}
  s_3s_2s_3 \\
  s_3s_2 \\
  s_3 \\
\end{array}
\]
Note I have changed notation from the book with these diagrams. For instance, $s_3 = 1243$ and $s_2s_3 = 1423$.

**Theorem 2.** If $w = s_1 \cdots s_q$ is reduced, then $u \leq w$ if and only if there exists a reduced expression $u = s_{i_1} \cdots s_{i_k}$ for $1 \leq i_1 < \cdots < i_k \leq q$.

**Corollary 3.** If $w = s_1 \cdots s_q$ is reduced, then $u \leq w$ if and only if every reduced expression for $w$ has a subword that is a reduced expression for $u$.

This theorem tells us exactly when $u \leq w$, but using an idea like the first Lemma, we can get even more precise about what things appear in between $u$ and $w$.

**Theorem 4** (Chain Property). For $u < w$ there exists a chain $u = x_0 < x_1 < \cdots < x_k = w$ such that $\ell(x_i) = \ell(u) + i$.

For instance, we know from Theorem 2 that $s_3 < s_1s_2s_3s_1$. Now we can follow one (there are multiple) chain from $s_3$ to $s_1s_2s_3s_1$.

\[
s_{2s_1s_2}s_3 = s_1s_2s_3s_1 = 4213
\]

\[
s_1s_2s_3 = 4123
\]

\[
s_2s_3 = 1423
\]

\[
s_3 = 1243
\]

**Proposition 5** (Lifting Property). Suppose $u \leq w$ and $s \in D_L(w) - D_L(u)$. Then, $u \leq sw$ and $su \leq w$.

Recall $T_L(w) = \{ t \in T : \ell(tw) < \ell(w) \}$, where $T$ is the set of reflections (conjugates to the simple reflections), and $D_L(w) = T_L(w) \cap S$, where $S$ is the set of simple reflections.

From our previous example, $s_1 \in D_L(w)$ for $w = s_1s_2s_3s_1$, but $s_1 \notin D_L(u)$ for $u = s_3$. We get the following picture.

\[
w = s_1s_2s_3s_1 = 4213
\]

\[
s_1w = s_2s_3s_1 = 2413
\]

\[
s_1u = s_1s_3 = 2143
\]

\[
u = s_3 = 1243
\]
The fact that $sw$ covers $su$ is a coincidence from our smallish example. These two need not even be comparable according to the Proposition.

**Corollary 6.**  
1. For $s \in S$, $t \in T$, $s \neq t$ where both $sw$ and $tw$ cover $w$, then $sw$ and $tw$ are covered by $stw$.

2. For $s, s' \in S$ where $sw$ and $ws'$ cover $w$, then either $sw$ and $ws'$ are covered by $sws'$ or $w = sws'$.

Here is one example of the first part of the corollary. Since the poset is graded by length, in order for $sw$ and $tw$ to cover $w$ (which means to be in the next highest grading?), either $t$ is also a simple reflection, or there will be some cancellation in $tw$.

\[
(s_1s_2s_3s_2s_1)(s_3s_1) = s_1s_2s_3s_1 = 3412
\]

\[
(s_1s_2s_3s_2s_1)(s_3s_1) = s_1s_2s_3 = 3142
\]

\[
s_2(s_3s_1) = 2413
\]

\[
s_3s_1 = 2143
\]

It is not too hard to find an example of one case of part two of the corollary.

\[
s_2s_3s_2 = 1432
\]

\[
s_3s_2 = 1342
\]

\[
s_2s_3 = 1423
\]

\[
s_3 = 1243
\]

If $s = s'$, such as $s = s' = s_1$ and $w = s_3$, then we have $sws' = w$, but I am not sure if there is a nice, general way to know when this happens.

**Question** For what $s, s' \in S$ and $w \in W$ do we have $sws' = w$?

## 2 2.3

There is one (well, multiple..) property we did not mention from last section. We will focus on this property, and its consequences for finite reflection groups, in this section.

**Proposition 7.** *Bruhat order is a directed poset.*
So, if our reflection group is finite, we will have a greatest element. This will be denoted \( w_0 \). We collect several of the properties of \( w_0 \) in the following (which combines several propositions and corollaries from the text).

**Proposition 8.** Let \( w_0 \) be the unique greatest element of \( W \).

- \( \Delta_L(w_0) = S \).
- \( w_0^2 = e \) where \( e \) is the identity of \( W \).
- \( w \leq w_0 \) for all \( w \in W \). (just the definition of the greatest element)
- \( T_L(ww_0) = T - T_L(w) \) and a similar statement holds for multiplying on the right.
- \( \ell(ww_0) = \ell(w_0) - \ell(w) = \ell(w_0w) \)
- \( \ell(w_0ww_0) = \ell(w) \) for all \( w \in W \).
- \( \ell(w_0) = |T| \)

Besides these basic properties, we also have that multiplication by \( w_0 \) can transform \( W \) in several ways.

**Proposition 9.** The following maps on \( W \), \( w \rightarrow ww_0 \) and \( w \rightarrow w_0w \) are antiautomorphisms, and \( w \rightarrow w_0ww_0 \) is an automorphism.

An antiautomorphism is a bijective map that reverses the order of multiplication. We will illustrate this proposition for the Bruhat order of \( S_3 \).

First, we have the original Bruhat order of \( S_3 \).

![Bruhat Diagram](image)

The following is the result of multiplying all elements on the right by \( w_0 = s_1s_2s_1 = s_2s_1s_2 \).

4
In the case of the symmetric group, multiplication on the right by \( w_0 \) should switch the places of the permutation. For instance, \( s_1 = 213 \), and \( s_1 w_0 = s_1 s_2 = 312 \).

Next, we multiply by \( w_0 \) on the left.

Multiplication on the left by \( w_0 \) should swap the values of the permutation. Again, for \( s_1 = 213 \), we now have \( w_0 s_1 = s_2 s_1 = 231 \). The value 2 stays fixed as it is in the middle of all possible values, and 1 and 3 swap.

Finally, we conjugate all elements by \( w_0 \).

So, \( w_0 s_1 w_0 = s_2 \) and \( w_0 s_2 w_0 = s_1 \), but the other elements are fixed. So we have only permuted the generators.
We may wonder what other automorphisms exist for a finite reflection group \( W \). The answer depends on the size of the generating set \( S \). It turns out \( |S| = 2 \), as in our example, is a special case. We will set it aside briefly, and cover the other larger case.

**Theorem 10.** Suppose \((W, S)\) is irreducible and \( |S| \geq 3 \). If \( \varphi : W \rightarrow W \) is an automorphism of the Bruhat order and \( \varphi(s) = s \) for all \( s \in S \), then either \( \varphi(x) = x \) for all \( x \in W \) or \( \varphi(x) = x^{-1} \) for all \( x \in W \).

**Corollary 11.** If \((W, S)\) is irreducible and \( |S| \geq 3 \), then the automorphism group of \( W \) is generated by the diagram automorphisms and \( x \rightarrow x^{-1} \).

In fact, we have that the group of automorphisms of a finite, irreducible Coxeter group of rank \( \geq 3 \) is either \( \mathbb{Z}/2\mathbb{Z} \) (for \( x \rightarrow x^{-1} \)) or \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), if \( x \rightarrow w_0 x w_0 \) is not the identity. This leaves the following interesting question.

**Question** For what finite Coxeter groups \( W \) is the automorphism \( \varphi(x) = w_0 x w_0 \) the identity?

The answer to this question (for rank at least 3) is provided by an exercise in chapter 4. I will state this result without proof for the time being, and hopefully prove it once I have the appropriate tools.

**Claim** The automorphism \( \varphi(x) = w_0 x w_0 \) is not the identity of \( W \) if and only if \( W \) is not \( A_n(n \geq 2) \), \( D_n \) (for \( n \) odd), \( E_6 \), or \( I_2(m) \) for \( m \) odd.

Note this does cover some rank 2 cases, namely \( A_2 \) and \( I_2(m) \) for odd \( m \). We already saw in our example of \( S_3 \) (which is also \( A_2 \) and \( I_2(3) \)) that this conjugation was not the identity on \( S \), our set of generators.

Now, to wrap up the other rank 2 cases, we have the following exercise.

**Exercise 2.2** Show that the automorphism group of the Bruhat order of \( I_2(p) \) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^{p-1}\).

**Proof.** The Bruhat order of \( I_2(p) \) looks like that of \( S_3 \), but with more “rungs”. That is, at each rank, there are two elements, and each connect with both elements at the next rank up (assuming the next rank is not the greatest, at which there is only \( w_0 \).) Therefore, we can always swap the two elements at the same rank and preserve the bruhat order. For \( I_2(p) \), there are \( p - 1 \) ranks where there are two elements. These correspond to words lengths \( 1, 2, \ldots, p - 1 \). Therefore, any automorphism can be mapped to an element of \((\mathbb{Z}/2\mathbb{Z})^{p-1}\) by sending it to a \( p - 1 \) tuple, where a 1 at location \( i \) signifies that we swapped the two elements of length \( i \), and a 0 signifies otherwise - we did not swap these. This is also injective, since the only thing mapping to the all 0 tuple is the identity automorphism. Therefore, since these two groups are the same size, we have an isomorphism. \( \square \)

I also will think about problem 14...