Exer 1.3.1: Show that all W s.t. rank(W) = 2 are dihedral groups.

Hint: Use Theorem 1.5: \( W = \langle s_\alpha \mid \alpha \in \Delta \rangle = \langle s_\alpha \mid \alpha \in \Phi \rangle \).

Pf. Since \( \text{rank}(W) = 2 \), every simple system \( \Delta \leq \Phi \) has exactly two elements. Then Theorem 1.5 says

\[ W = \langle s_\alpha \mid \alpha \in \Delta \rangle = \langle s_\alpha, s_\beta \rangle \]

for \( \Delta = [\alpha, \beta] \). WLOG, we can view this as a system in \( \mathbb{R}^2 \) and under some basis \( \alpha = (1, 0) \).

Then \( \beta = k (\cos \theta, \sin \theta) \) for \( \theta \in (0, \pi) \).

We can also take \( k = 1 \), since scalar multiples will define the same line. Since \( W \) must have finite order, \( \theta \) must be \( \frac{\pi}{m} \) for some positive integer \( m \).

Thanks to Craig for figuring out the finite order part via reflecting across \( \alpha, \alpha + \beta \), etc. 😊
Exer 1.3.2: Find simple systems for \(I_2(m), A_{m-1}, B_n, \) and \(D_n.\)

\[\star I_2(m) : \text{Given in Exer 1.3.1.}\]

\[\Delta = \{ (0, 1), (\cos \theta, \sin \theta) \} \text{ for } \theta = \frac{\pi}{m}.\]

\[\star A_{m-1} : \Delta = \{ (0, \ldots, 0, 1, -1, 0, \ldots, 0) \mid 1 \leq i \leq m-1 \} \]

\[\star B_n. \text{ Roughly, we want the } \Delta \text{ from } A_{m-1}, \text{ which we denote } \Delta_A, \text{ unioned with a single sign swap.}\]

\[\rightarrow \Delta_A = \Delta_A \cup \{ e_i \} \]

\[\star D_n. \text{ Similarly } \Delta = \Delta_A \cup \{ (e_1 - e_2) \rightarrow (e_1 + e_2) \}.\]

Thanks to Eric who fixed an error in \(\Delta_A\).