Notes and Exercises from Humphrey’s

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1 Section 1.2

This section introduces root systems.

A root system $\Phi$ is a finite set of nonzero vectors satisfying:

- $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$
- $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

2 Section 1.5

The big theorem of this section gives us that a finite reflection group, call it $W$, is generated by simple reflections (reflections of simple roots for a given simple system).

We also get a useful corollary that, given a simple system the $W$ orbit of any root meets this simple system.

Exercise 1 Let $\Phi$ be a root system of rank $n$ consisting of unit vectors. If $\Psi \subset \Phi$ is a set of $n$ roots whose mutual angles agree with those between the roots in some simple system, then $\Psi$ must be a simple system.


Proof. I am assuming that our reflection group is essential, so that the $\Phi$ spans the entire vector space we live in. Let $\Phi$ be a root system with simple system $\Delta$ and positive system $\Pi$. I will use some a’s, b’s instead of $\alpha$ and $\beta$ because I am lazy.

Let $\Delta = \{a_1, \ldots, a_n\}$ be a simple system, and let $\Psi = \{b_1, \ldots, b_n\}$ be another subset of $\Phi$, the root system, indexed such that the angle between $a_i$ and $a_j$ is the same as the angle between $b_i$ and $b_j$. Since these are all unit vectors, and we are in a Euclidean space using the dot product as our symmetric bilinear form, it follows that $(a_i, a_j) = (b_i, b_j)$ for all pairs $i, j$. 

An important part of this proof appears to be showing that we can construct an orthogonal transformation $T$ to go between $\Delta$ and $\Psi$. Define $T(a_i) = b_i$. Then, for all pairs $i, j$, the nice hypotheses give us that $T$ is orthogonal,

$$(Ta_i, Ta_j) = (b_i, b_j) = (a_i, a_j).$$

Now, since $\Delta$ is a simple system of $\Phi$, we have that $T\Delta = \Psi$ is a simple system of $T\Phi = \Phi'$. We would like to show that $\Phi = \Phi'$. We can get most of the way there by showing that $W = W'$, where $W$ is the reflection group associated to $\Phi$ and similarly for $W', \Phi'$.

By Theorem 1.5, a finite reflection group is generated by simple reflections. It follows that,

$$W' = \langle s_{b_i} : b_i \in \Psi \rangle \subseteq \langle s_{a_i} : a_i \in \Delta \rangle = W$$

Moreover, $W' = \langle s_{b_i} : b_i \in \Psi \rangle = \langle s_{Ta_i} : a_i \in \Delta \rangle = \langle Ts_{a_i}T^{-1} : a_i \in \Delta \rangle$. This last equality comes from Proposition 1.2. Rearranging, we have that $W' = T\langle s_{a_i} : a_i \in \Delta \rangle = TW'T^{-1}$. Since these reflection groups are conjugate, they have the same cardinality. But $W' \subseteq W$ as well, so $W = W'$.

Now, $\Phi' = W'\Psi$ since $\Psi$ is the associated simple system. From the fact that $W = W'$, we can say more,

$$\Phi' = W\Psi \subseteq W\Phi = \Phi$$

However, $\Phi' = T\Phi$, so again these root systems should have the same cardinality. Thus, $\Phi' = \Phi$, and it follows that $\Psi$ was a valid simple system of $\Phi$ all along!

Exercise 2 Given a simple system $\Delta$, no proper subset of the simple reflections can generate $W$.

Proof. Let $\Phi$ be a root system with simple system $\Delta$ and $W$ the associated reflection group. Suppose for sake of contradiction that $\alpha \in \Delta$ such that we do not need $s_\alpha$ as a generator of $W$. By corollary 1.5, there exists $w \in W$ such that $w(-\alpha) \in \Delta$. However, by Proposition 1.4, for a simple $\beta$, $s_\beta$ sends all positive roots to positive roots, except for $\beta$. Then, similarly, $s_\beta$ will preserve all negative roots except for $-\beta$. If there is no term $s_\alpha$ in $w$, then $-\alpha$ will never become positive, so we have no hope for $-\alpha$ to land in $\Delta$. Very sad!

Exercise 3 If $\beta \in \Pi \setminus \Delta$, then $ht(\beta) > 1$.

Proof. Let $S = \{\beta \in \Pi \setminus \Delta : ht(\beta) \leq 1\}$. We would like to show that $S$ is empty. Suppose for sake of contradiction $S$ is nonempty. Since our root systems are finite, we can certainly find an element of $S$ of minimal height. Call this $\gamma$. Decompose $\gamma$ in terms of $\Delta$, $\gamma = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Since $\gamma$ is positive, all $c_\alpha$ are positive. Therefore, $0 \leq c_\alpha \leq 1$. Pick $\alpha'$ such that $c_\alpha' > 0$. Then, $s_{\alpha'} \gamma \in \Pi$ still by Proposition 1.4. However, $s_{\alpha'} \gamma$ was obtained by subtracting a multiple of $\alpha'$ from the decomposition of $\gamma$. This contradicts the claim that $\gamma$ had minimal height in $S$. Very sad!
The theorem of this section gives that the permutation action of $W$ is “simply transitive”. That is, if a reflection fixes a simple system, it must be the identity reflection.

A reflection of particular interest is the one that swaps the positive and negative systems. This is a reflection of maximal length, since it sends a maximal number of positive roots to negative roots and vice versa. It is often called the “longest element” and has many interesting properties!

**Exercise 1** What is the longest element in $S_n$, relative to the simple system $\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n$.

**Proof.** With the simple system, the generators of $S_n$ are the transpositions $s_i = (i, i+1)$.

I claim the longest element is of the form

$$s_1s_2\cdots s_{n-2}s_{n-1}s_{n-2}\cdots s_2s_1\cdots s_{n-2}\cdots s_3s_2s_3\cdots$$

*** I should have written this in reverse order, though this order probably also works?

the end depends on whether $n$ is odd or even. If $n = 2k + 1$, the final terms are $s_ks_{k+1}s_k$. If $n = 2k$, the final terms are $s_{k-1}s_ks_{k+1}s_{k-1}s_k$. This was found based on some previous intuition, some playing around with small examples, and the fact that the length will be the same as the cardinality of the positive roots. In $S_n$, these are all $\epsilon_i - \epsilon_j$ for $i < j$. There are $\binom{n}{2}$ such positive roots. For the above defined positive root, the first peak has $n - 1$ terms, from $s_1$ to $s_{n-1}$, then the corresponding dip(?) has $n - 2$ terms, from $s_{n-2}$ to $s_1$. Then, the second peak, if it exists, from $s_2$ to $s_{n-2}$ has $n - 3$ terms, and so on. So the length of this term is $\sum_{i=1}^{n-1} i = \binom{n}{2}$, as desired. (Note this means the negative roots are exactly $\epsilon_j - \epsilon_i$ for $i < j$.)

This of course is not enough to verify that this is a valid longest word. We either need to make sure this word is reduced, that is it truly has this length, or to verify that it has the property of the longest word, which is that it sends all positive roots to negative roots. We will explore this second option.

Given $\epsilon_i - \epsilon_j$, first suppose that $i = 1$. Then, $s_1(\epsilon_1 - \epsilon_j) = \epsilon_2 - \epsilon_j$, and each subsequent simple reflection with increase the index on the first term while fixing the other term. Then, $s_{j-1}(\epsilon_{j-1} - \epsilon_j) = \epsilon_j - \epsilon_{j-1}$. If $j = n$, then the next reflections will decrease the index on the second term, until $s_1(\epsilon_j - \epsilon_2) = \epsilon_j - \epsilon_1$. Note that $s_1$ does not appear again, so this second term will always be $\epsilon_1$, and no matter what the next reflections do to the first term, this will always be a negative root. If $j \neq n$, then there are more reflections, $s_j, s_{j+1}, \ldots$ which will increase the index on the first term above until we have $\epsilon_n - \epsilon_j$. Since $s_{n-1}$ will not appear again in this word, the first term is now fixed at $n$, so again this will be a negative root.

Now, if $1 < i < j - 1 < n$, then in the first peak and dip, we have that $s_i(\epsilon_i - \epsilon_j) = \epsilon_{i-1} - \epsilon_j$, then $s_{j-1}(\epsilon_{i-1} - \epsilon_j) = \epsilon_{i-1} - \epsilon_{j-1}$. The next reflections, up to the peak and
back will leave this unchanged. Then, 

\( s_{i-1} \cdots s_{j-1}(\epsilon_{i-1} - \epsilon_{j-1}) = \epsilon_i - \epsilon_j \).

Therefore, if the indices of the peak and dip are far from the indices of your positive root, this cycle will leave the root unchanged. However, our cycles get smaller and smaller, so at some point, either the lower index will match the smallest index in a cycle, or the larger index will match the largest index in a cycle (or both!). And, copying what was stated in the previous paragraph, in this cycle the root will permanently become negative.

If we have a simple root, \( i = j - 1 \), then we again need to wait until we get to a cycle whose largest or smallest indexed reflection matches one of the indices. The action will look a little different from described above, but is similar idea.

**Exercise 2** In any reduced expression for the longest element, \( w_\alpha \), every simple reflection must occur at least once.

**Proof.** We know that the simple roots are a subset of the positive roots. Moreover, we can write \( w_\alpha \) in terms of simple reflections \( s_\alpha (\alpha \in \Delta) \) for any simple system \( \Delta \). Moreover, by Proposition 1.4, \( s_\alpha \) fixes \( \Pi \smallsetminus \{\alpha\} \); namely, \( s_\alpha \) leaves all other simple roots from this system fixed. Therefore, if we want to make all simple roots negative, we need at least one copy of \( s_\alpha \) for all \( \alpha \in \Delta \).

**4 1.11**

We define the Poincare polynomial of \( W \) to be

\[
W(t) := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{\ell(w)}
\]

where \( \ell \) denotes the length, and \( a_n := |\{w \in W : \ell(w) = n\}| \).

We can use the following formula to inductively compute \( W(t) \).

\[
\sum_{I \subseteq S} (-1)^I \frac{W(t)}{W_I(t)} = \sum_{I \subseteq S} (-1)^I W^I(t) = t^N
\]

Note that \( W_I(t) \) is the reflection group generated by \( I \subseteq S \), where \( S \) is the set of simple reflections associated to a simple system \( \Delta \). Then, \( W^I := \{w \in W : \ell(ws) > \ell(w) \forall s \in I\} \). (Orthogonalish?) The reflections in \( W^I \) only deal with reflections of roots not generated by the roots in \( I \). Does a decomposition of \( W \) follow?

**Exercise 1** When \( W = S_3 \), use the formula in the proposition to compute \( W(t) \) inductively, starting with the fact that \( W(t) = 1 + t \) for a group of rank 1. Do the same for \( D_m \) in general.

**Proof.** If \( W = S_3 \), then any simple system is rank 2, so has two generators, so \( S \), our generating set of \( W \), is also size 2. The corresponding subsets of \( S \) are the empty set,
two subsets size 1, and the whole set. This gives
\[ W(t)((-1)^0 + (-1)^1/(1 + t) + (-1)^1/(1 + t)) + (-1)^2 = t^3 \]
we rearrange and clean up to get,
\[ W(t) = \frac{(t^3 - 1)(1 + t)}{t - 1} = (t^2 + t + 1)(t + 1) = t^3 + 2t^2 + 2t + 1 \]
by the original interpretation of \(W(t)\), we see that the coefficient on the largest
degree term and the constant term correspond to the longest element and the identity
element, respectively. Then, the coefficient of 2 on \(t^2\) corresponds to \(s_1s_2\) and \(s_2s_1\),
and the coefficient of 2 on \(t\) correspond to the generators, \(s_1\) and \(s_2\).

We can do a very similar computation for \(D_m\) since all dihedral groups have two
generators. The only thing that will change is the \(N\) in the above proposition. We
have that \(N\) should be the cardinality of the positive root system. In this text \(D_m\) is
the group of transformations of an \(m\)-gon, so is size \(2m\). Therefore, the root system is
size \(2m\), so that there are \(m\) positive roots. The same computation as above results in
\[ \frac{(t^m - 1)(1 + t)}{t - 1} = (t^{m-1} + t^{m-2} + \cdots + 1)(t + 1) = t^m + 2t^{m-1} + 2t^{m-2} + \cdots + 2t + 1. \]

Exercise 2 Notice the identity we get if we substitute 1 for \(t\) in the above equation,
\[ \sum_{I \subseteq S} (-1)^{|I|} \frac{|W|}{|W_I|} = 1 \]
This identity permits an inductive calculation of \(|W|\) when \(|S|\) is odd. Suppose
for example that \(|S| = 3\) and that the dihedral subgroups \(W_I\) are of respective orders
4, 6, 10. What is \(|W|\).

Proof. If \(S\) is size 3, then there is one subset size 0, three subsets size 1, three subsets
size 2, and one subset size 3. We are provided information on the subgroups of size
2, and the rest of the sizes are straightforward. We get the following equation,
\[ |W|(1 - 3\left(\frac{1}{2}\right) + \frac{1}{4} + \frac{1}{6} + \frac{1}{10}) - 1 = 1 \]
and rearranging, we have \(|W| = 120. \]
This formula is only useful if the size of the generating set is odd. Note we will
always have a term of the form \((-1)^{|S|} \frac{|W|}{|W_S|} = (-1)^{|S|}\). If \(|S|\) is even, then this is 1,
and we can subtract 1 from each side of the equation to get \(|W| \cdot C = 0\), where \(C\)
comes from all the other terms in the sum. Thus, we cannot determine the size of \(W\)
with this formula if the generating set is of even size.