Exercise (Humphreys 2.10). For each of the crystallographic root systems, write the highest root $\tilde{\alpha}$ as a $\mathbb{Z}$-linear combination of the simple roots.

In what follows, let $e_1, e_2, \ldots, e_k$ be the standard basis of $\mathbb{R}^k$. For all types except A and E, the number $k$ is the rank of the Coxeter system; in type A it is $n + 1$ and for type E it is 8.

We omit the linear algebra used to solve the problems, because it is trivial in the following sense: for each type, there is a coordinate of the highest root which is nonzero for only one simple root, which determines its coefficient in linear combination, and then we proceed recursively. The order in which we have written the roots in the linear combinations respects this recursive solution.

**Type A:** Recall that the simple roots are $\alpha_i = e_i - e_{i+1}$ for any $1 \leq i \leq n$, and the highest weight root is $\tilde{\alpha} = e_1 - e_{n+1}$. Therefore

$$\tilde{\alpha} = \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_{n-1} + \alpha_n.$$ 

**Type B:** Recall that the simple roots $\alpha_i$ are the same as in type A for any $1 \leq i \leq n - 1$, and $\beta = e_n$; and that the highest weight root is $\tilde{\alpha} = e_1 + e_2$. Therefore

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + 2\beta.$$ 

**Type C:** Recall that the simple roots are the same as in type B, except that $\beta = 2e_n$; and that the highest weight root is $\tilde{\alpha} = 2e_1$. Therefore

$$\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_{n-1} + \beta.$$ 

**Type D:** Recall that the simple roots $\alpha_i$ are the same as in type A (i.e. $\alpha_i = e_i - e_{i+1}$) for any $1 \leq i \leq n - 1$, and $\delta = e_{n-1} + e_n$; and that the highest weight root is $\tilde{\alpha} = e_1 + e_2$. Therefore

$$\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \delta.$$ 

Now the fun begins...

**Type E:** Recall that the simple roots are $\omega = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$ and $\epsilon_1 = e_1 + e_2$, together with the roots $-\alpha_i$ from type A, for $1 \leq i \leq n - 2$ (where $n$ is either 6, 7, or 8). Note in particular that, regardless of $n$, all of the roots live in $\mathbb{R}^8$.

[Careful: this notation differs rather sharply from Humphreys. For $i \geq 3$, his $\alpha_i$ is our $-\alpha_{i-1}$.] 

The highest roots do not admit a uniform description, so we handle each case separately. The calculations are carried out on the next page.
For $E_6$ we have $\tilde{\alpha} = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$, and so

$$\tilde{\alpha} = \omega + (-\alpha_4) + 2(-\alpha_3) + 3(-\alpha_2) + 2\epsilon_1 + 2(-\alpha_1).$$

Note that the last two terms do not follow the recursive framework, but the linear algebra is fairly straightforward since by considering the first coordinate we see that the last two terms must have the same coefficient.

For $E_7$ we have $\tilde{\alpha} = e_8 - e_7$, and so

$$\tilde{\alpha} = 2\omega + (-\alpha_5) + 2(-\alpha_4) + 3(-\alpha_3) + 4(-\alpha_2) + 3\epsilon_1 + 2(-\alpha_1).$$

Again the last two terms do not follow the recursive framework: the equations that determine these coefficients are $c_\alpha + c_\epsilon = 5$ and $c_\alpha - c_\epsilon = 1$.

For $E_8$ we have $\tilde{\alpha} = e_8 + e_7$, and so

$$\tilde{\alpha} = 2\omega + 2(-\alpha_6) + 3(-\alpha_5) + 4(-\alpha_4) + 5(-\alpha_3) + 6(-\alpha_2) + 4\epsilon_1 + 3(-\alpha_1).$$

The equations that determine the last two coefficients are $c_\alpha + c_\epsilon = 7$ and $c_\alpha - c_\epsilon = 1$.

And now for the victory lap...

**Type F:** Recall that the simple roots are $\psi = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ as well as $\alpha_2$, $\alpha_3$, and $\beta = e_4$ as in type B; and that the highest root is $\tilde{\alpha} = e_1 + e_2$. Therefore

$$\tilde{\alpha} = 2\psi + 2\alpha_2 + 3\alpha_3 + 4\beta.$$  

**Type G:** Recall that the simple roots are $\chi = -2e_1 + e_2 + e_3$ and $\alpha_1$ as in type A; and that the highest root is $2e_3 - e_1 - e_2$. Therefore

$$\tilde{\alpha} = 2\chi - 3\alpha_1.$$  

**Commentary:** I do not have any particular insight into these calculations. Perhaps a useful followup would be to prove that, say, the given highest root of $E_6$ was actually the highest root of $E_6$. Presumably this could be done in an equally unenlightening way, but more optimistically, it might give some intuition about the structure of the root poset.