Section 1.3: Positive and Simple Root Systems

In these notes, we will use the standing assumption that $\Phi$ is a root system on a (real) Euclidean space $V$. We let $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$.

**Question.** Can we “simplify” our root system to get rid of dependencies (while retaining the structure of the original)?

We will begin by reminding the reader that we can always construct a total ordering on $V$ by ordering a basis for $V$ and then extending via lexicographic ordering. Let $<_V$ be a total ordering on $V$. We say that $v \in V$ is positive if $v >_V 0$. Recall that the sum of positive vectors is positive, as is a positive scalar multiple of a positive vector.

We will use this fact to partition our root system into positive and negative roots. Note that all roots must be either strictly positive or strictly negative since none has length 0.

**Definition** (Positive System). Let $\Pi \subseteq \Phi$ be a subset of roots that are all (strictly) positive with respect to some total ordering on $V$.

**Definition** (Negative System). Let $-\Pi \subseteq \Phi$ be a subset of roots that are all (strictly) negative with respect to some total ordering on $V$.

**Example 1.** Let $\alpha = (1, 0)$ and let $\Phi = \{\alpha, -\alpha\}$. Then $\Pi = \{\alpha\}$ and $-\Pi = \{-\alpha\}$.

Since roots come in $\pm$ pairs, we have $\#\Pi = \#(-\Pi)$ and $\Phi = \Pi \cup -\Pi$.

**Definition** (Simple System). Let $\Delta \subseteq \Phi$. We say that $\Delta = \{\beta_1, \beta_2, \ldots\}$ is a simple system if it satisfies the following two conditions:

1. $\Delta$ is a vector space basis for $\text{span}_\mathbb{R} \Phi \subseteq V$, and
2. Each $\alpha \in \Phi$ can be expressed as a linear combination of $\beta_i$ from $\Delta$ AND all the coefficients are either all positive or all negative.

The following two theorems show that simple systems exist.

**Theorem 1** (1.3(a)). Let $\Delta$ be a simple system in $\Phi$. Then there exists a positive system $\Pi$ such that $\Delta \subseteq \Pi$.

**Theorem 2** (1.3(b)). Every positive system $\Pi$ contains a unique simple system $\Delta \subseteq \Pi$. 
\[ \beta = (\cos \frac{\pi}{m}, \sin \frac{\pi}{m}) \]
\[ \alpha = (1, 0) \]

Figure 1: A simple system for \( I_2(m) \).

We have yet to show how simple systems relate to the structure of \( W \) (the reflection group generated by the \( s_\alpha \)’s where \( \alpha \in \Phi \)). In Section 1.5, we’ll show that \( W \) can be generated by just the \( s_\alpha \)’s where \( \alpha \in \Delta \) where \( \Delta \) is ANY simple system of \( \Phi \).

**Corollary.** Let \( \Delta \subseteq \Phi \) be a simple system. Then for all distinct \( \alpha, \beta \in \Delta \) we have \( (\alpha, \beta) \leq 0 \).

Since the number of elements in \( \Delta \) is the dimension of the (real) span of \( \Phi \), the number of elements in a simple system is invariant under choice of \( \Delta \). That is: for any simple system \( \Delta \subseteq \Phi \), the number of elements in \( \Delta \) is the same.

**Definition** (Rank). The rank of a root system \( \Phi \) is the number of elements in a simple system. Equivalently the rank of \( W \) is the number of elements in \( \Delta \subseteq \Phi \).

**Example 2.** Let us consider the dihedral group \( I_2(m) = D_m \). One simple system is \( \alpha = (1, 0) \) and \( \beta = (\cos \frac{\pi}{m}, \sin \frac{\pi}{m}) \). Then rank(\( I_2(m) \)) = 2 for all \( m \). See Figure 1.

**Example 3.** The symmetric group can be generated by the \( n - 1 \) transpositions \((i, i + 1)\) for \( i \in \{1, 2, ..., n - 1\} \). Then rank(\( A_{n-1} \)) = \( n - 1 \).