1. Evaluate the integral

\[
\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy
\]

Sketch the region of integration. You should be able to do this by hand, but here is what the region looks like, as plotted by mathematica.

This is somewhat difficult to see, so here is a cross section.
Although these pictures aren’t bad, they are not a particularly precise description. A better way of describing the region would be in equations. This region is the volume between $z = x^2 + y^2$ and $z^2 = x^2 + y^2$ bounded by the planes $y = 0$, $x = 1$ and $z = 1$.

It is often difficult to know what coordinate system to use, the two obvious choices here cylindrical and spherical. In general if a region has “messy” z-coordinates, its a good bet to try cylindrical coordinates.

To describe the region in cylindrical coordinates we say this is the volume between $z = x^2 + y^2$ and $z^2 = x^2 + y^2$ bounded by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$.

We then setup the integral as

$$\int_{0}^{1} \int_{0}^{\sqrt{x^2+y^2}} \int_{0}^{\frac{1}{2}} r r^2 \cos(\theta) r \sin(\theta) dz d\theta dr$$

If you work this integral all the way through you end up with an answer of

$$\frac{1}{96}$$
2. Find the volume of the region between a sphere with radius 1 and a sphere with radius 2, in the region of $\mathbb{R}^3$ bounded by $x \geq 0, y \geq 0, z \geq 0$

This was meant to be a fairly simple problem, you should get practice reading a word problem, and thinking in spherical coordinates from this problem.

The bounds $x \geq 0, y \geq 0, z \geq 0$ tell us that the region is entirely located in the section of the space with $x, y, z$ all non-negative. So the region should bet the “shell” of a sphere.

It will look like this

![3D graph](image)

To parameterize this region we first note that $\rho$ can be anything between 1 and 2. Since this is all in the positive section of $\mathbb{R}^3$ we know that the most $\theta$ can be is $\frac{\pi}{2}$ and we likewise know that $\phi$ can be at most $\frac{\pi}{2}$

This give the following bounds on our variables

$$0 \leq \theta \leq \pi/2$$
$$0 \leq \phi \leq \pi/2$$
$$1 \leq \rho \leq 2$$

The integral is then

$$\int_{\theta=0}^{\theta=\pi/2} \int_{\phi=0}^{\phi=\pi/2} \int_{\rho=1}^{\rho=2} \rho^2 \sin \theta \, d\rho \, d\phi \, d\theta$$
\[ \int_0^2 \int_0^\pi \rho^2 \sin(\phi) \, d\phi \, d\rho \]

Which we can evaluate to give a volume of \( \frac{7\pi}{6} \)

3. Evaluate the integral over the region \( R \) which is bounded by the lines \( y = x, y = x - 2, y = -2x, y = 3 - 2x \)

\[ \int_R (3x + 4y) \, dA \]

Use the linear transformation \( x = 1/3(u + v) \) and \( y = 1/3(v - 2u) \) to evaluate the integral.

For the sake of convincing you this isn’t an integral we’d want to do by hand, here is a picture of the region we are integrating over

You will note that in order to do this integral without changing coordinates, we would need to setup 3 different integrals, each with different bounds. Yech!

We can however take from this picture the coordinates of the “corners” of the parallelogram. They will be

\[ (0, 0), (1, 1), \left(\frac{2}{3}, \frac{-4}{3}\right), \left(\frac{5}{3}, \frac{-1}{3}\right) \]

These coordinates are in the \( x - y \) plane and we want to find what the preimage of the map is in the \( u - v \) plane. You could solve these equations for \( u \) and \( v \) explicitly, but I will use some linear algebra in order to do it.
We know that for a point \((u, v)\) in the \(u-v\) plane that:

\[
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

Lets name this matrix

\[
M = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{2}{3} & \frac{1}{3}
\end{pmatrix}
\]

Expanding this equation out gives us exactly the transformation we are given in the problem statement, and gives us points in the \(x-y\) plane. But we currently have points in the \(x-y\) plane so we want to invert this mapping. We do this by taking the inverse matrix.

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^{-1} = \frac{1}{(ad-bc)} \begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix}
\]

In our specific case this gives our inverse being:

\[
A = \begin{pmatrix}
1 & -1 \\
2 & 1
\end{pmatrix}
\]

Now applying this matrix to the points we found earlier we get the new set of points.

\[(0,0)(0,3)(2,0)(2,3)\] (1)

So we recognize this as a rectangle. We would like to now setup the integral but first we have to change the integrand.

\[3x + 4y = \frac{1}{3}(-5u + 7v)\]

which we do simply by using the given equations. This allows us to setup the integral (in the \(u,v\) coordinates)

\[\int_0^2 \int_0^3 \frac{1}{3} (-5u + 7v) \det M \, du \, dv = \frac{11}{3}\]

Which is the same answer we would have gotten had we just blindly integrated. In this example it seems like we are doing a lot more work, but if you are comfortable with these sorts of change of variable problems, you can often save a lot of work this way. Its worthwhile going and looking through the book and making sure you see how I’ve used each of theorems, and how the regions are related.

Consider the integral

\[
\int_0^3 \int_y^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-y^2}} (y+z) \, dz \, dx \, dy
\]
4. Find a parametric equation for the cone $x^2 + y^2 - z^2 = 0$

(a) Find a downward facing normal vector for an arbitrary point on the cone. Is the cone a smooth surface?

A parametric equation for a cone is:

$$
\begin{align*}
  x(\theta, r) &= r \cos(\theta) & 0 \leq \theta \leq 2\pi \\
  y(\theta, r) &= r \sin(\theta) & 0 \leq r \leq 2 \\
  z(\theta, r) &= r
\end{align*}
$$

We can verify that this is indeed the parametric equation for a cone by computing

$$
\begin{align*}
  x(\theta, r)^2 + y(\theta, r)^2 - z(\theta, r)^2 &= r^2 \cos^2(\theta) + r^2 \sin^2(\theta) - r^2 \\
  &= r^2 \cos^2(\theta) + r^2 \sin^2(\theta) - r^2 \\
  &= r^2 - r^2 = 0
\end{align*}
$$

(b) Find a downward facing normal vector.

Find a downward facing normal vector for an arbitrary point on the cone. Is the cone a smooth surface?

To compute the normal vector we take the parametric equations of the cone, as given above and compute the normal. That is we compute

$$
\left( \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r} \right) \times \left( \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right)
$$

As usual we setup the cross product as a determinant.

$$
\begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\
  \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta}
\end{vmatrix} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  \cos(\theta) & \sin(\theta) & 1 \\
  -r \cos(\theta) & r \cos(\theta) & 0
\end{vmatrix} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  -r \cos(\theta) & -r \sin(\theta) & r \cos^2(\theta) + r \sin^2(\theta) \\
  -r \cos(\theta) & -r \sin(\theta) & r
\end{vmatrix}
$$

In order to be a smooth surface, the normal vector must be non-zero everywhere. In this situations, the normal vector is zero when $r = 0$ so the cone is NOT a smooth surface.

In order to verify that the normal vector is downward facing we simply have to check that the $z$ component of the normal vector is negative. As our normal vector does not have a negative $z$ component we must have taken the cross product in the wrong order. A corrected version gives normal vector

$$
(r \cos(\theta), r \sin(\theta), -r)
$$

which is downward facing.
(c) Find the tangent plane at the point \((-1, 0, 1)\).

Find the tangent plane at the point \((-1, 0, 1)\)

To find the tangent plane we use the equations that we have been using all semester. That is for a plane with normal vector \(\hat{n}\) containing the point \((a, b, c)\)

\[
T(x, y, z) = \hat{n}(x - a, x - b, x - c)
\]

Here this formula gives us

\[-r(z - 1) - s(x + 1) \cos(\theta) - ry \sin(\theta)\]

Which is the equation for the tangent plane.

5. Let \(S\) denote the closed cylinder with bottom given by \(z = 0\) and top given by \(z = 4\) and the lateral surface given by \(x^2 + y^2 = 9\). Orient \(S\) without outward normals. Determine the Surface Integral

\[
\iint_S y\,dS
\]

(a) Is this a vector or a scalar surface integral?

The function \(F(x, y, z) = y\) is a scalar valued function, so we are computing a scalar surface integral.

(b) Over what surface are we integrating? Can you write it with parametric equations?

We are integration over the cylinder. We can write it with parametric equations, we need three different equations in order to write it, one for the top of the cylinder, one for the bottom, and one for the lateral surface.

We will use cylindrical coordinates for all of them.

Here is the top of the cylinder, it is just a disc of radius 3, located at a fixed \(z = 4\)

\[
\begin{align*}
x(\theta, r) &= r \cos(\theta) & 0 \leq \theta \leq 2\pi \\
y(\theta, r) &= r \sin(\theta) & 0 \leq r \leq 3 \\
z(\theta, r) &= 4
\end{align*}
\]

Likewise the bottom of the cylinder will be a disc of radius 3 located at a fixed \(z = 0\)

\[
\begin{align*}
x(\theta, r) &= r \cos(\theta) & 0 \leq \theta \leq 2\pi \\
y(\theta, r) &= r \sin(\theta) & 0 \leq r \leq 3 \\
z(\theta, r) &= 0
\end{align*}
\]

And the lateral surface will be given by

\[
\begin{align*}
x(\theta, z) &= 3 \cos(\theta) & 0 \leq \theta \leq 2\pi \\
y(\theta, z) &= 3 \sin(\theta) & 0 \leq z \leq 4 \\
z(\theta, z) &= z
\end{align*}
\]
(c) Find a outward facing normal vector

We compute this the usual way, by taking a cross product of the partial derivatives. Since there are three parametric equations, we need three normal vectors

Computing the normal for the bottom section which we will call \( \mathbf{N}_1 \):

\[
\det \begin{pmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k}
\end{pmatrix}
\begin{pmatrix}
\cos(\theta) & \sin(\theta) & 0 \\
-r \sin(\theta) & r \cos(\theta) & 0
\end{pmatrix}
= (0, 0, 1)
\]

Computing the normal for the bottom section which we will call \( \mathbf{N}_2 \):

\[
\det \begin{pmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k}
\end{pmatrix}
\begin{pmatrix}
-r \sin(\theta) & r \cos(\theta) & 0 \\
\cos(\theta) & \sin(\theta) & 0
\end{pmatrix}
= (0, 0, -1)
\]

Computing the normal vector for the lateral sides which we will call \( \mathbf{N}_3 \):

\[
\det \begin{pmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k}
\end{pmatrix}
\begin{pmatrix}
-\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}
= (3 \cos(\theta), 3 \sin(\theta), 0)
\]

(d) Compute the integral

Now it is easy to compute the surface integral. We first compute the magnitude of each of the vectors

\[
\| \mathbf{N}_1 \| = 1 \\
\| \mathbf{N}_2 \| = 1 \\
\| \mathbf{N}_3 \| = 9
\]

So the surface line integral becomes

\[
\int_0^{2\pi} \int_0^4 9 \cos s ds \,dt + 2 \int_0^{2\pi} \int_0^1 t^2 \cos s ds \,dt = 0 + 0 = 0
\]

6. Here is a parametric equation for another surface.

\[
x(\theta, r) = \cos(\theta) \sin(\phi) \quad 0 \leq \theta \leq 2\pi \\
y(\theta, r) = \sin(\theta) \sin(\phi) \quad 0 \leq \phi \leq \pi \\
z(\theta, r) = \sin(\phi)
\]

(a) Is the surface smooth?

Our idea of smooth is that the normal vector is never zero on the surface. So we compute the normal vector
\[ \hat{n} = \det \left( \begin{array}{ccc} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} & \frac{\partial z}{\partial \theta} \end{array} \right) = \]

\[ = \det \left( \begin{array}{ccc} i & j & k \\ -\sin(\phi)\sin(\theta) & \cos(\theta)\sin(\theta) & 0 \\ \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi) \end{array} \right) = \]

\[ = (\cos(\theta)\sin^2(\theta), -\sin^2(\phi)\sin(\theta), -\cos(\phi)\sin(\phi)) \]

Without too much work you can verify that this is always non-zero.

(b) Find the tangent plane at the point \((\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}/2}{2}, \frac{1}{\sqrt{2}})\)

In order to find the tangent plane at this point we first must compute the what \(\theta, \phi\) values give us this point. Such values are \(\theta = \frac{\pi}{4}\) and \(\phi = \frac{\pi}{4}\). We will need these to compute the normal vector.

We have already calculated the normal vector at a general point, so we simply need to evaluate the equation we found above.

\[ N\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \left(\frac{-1}{4}, \frac{-\sqrt{3}}{4}, \frac{-1}{2}\right) \]

Now we recall the equation for a plane through a point with a specified normal vector. This gives us the equation

\[ z = \frac{1}{2} \left( 2\sqrt{2} - x - \sqrt{3}y \right) \]

(c) Find the area of the surface when \(0 \leq \theta \leq \frac{\pi}{4}\) and \(0 \leq \phi \leq \frac{\pi}{4}\)

To compute the area we setup the integral

\[ \int_0^{\pi/4} \int_0^{\pi/4} \| N(s, t) \| d\phi d\theta \]

Crunching through this integral gives us that a result of \(\frac{\pi^2}{16}\).