1. Find the second order Taylor expansion about the point \((0, 0)\) of the function

\[ f(x, y) = e^{xy} \]

We begin by computing the matrix of partial derivatives of \(f\).

\[ Df(x, y) = (e^{xy} y, e^{xy} x) \]

From this we compute the Hessian matrix

\[ Hf(x, y) = \begin{pmatrix} e^{xy} y^2 & e^{xy} y x + e^{xy} x y \\ e^{xy} y x + e^{xy} x y & e^{xy} x^2 \end{pmatrix} \]

Then we evaluate at the point \((0, 0)\) and find

\[ Df(0, 0) = (0, 0) \]
\[ Dg(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

Now we put these together to compute the degree 2 Taylor polynomial

\[ T_2(x, y) = f(0, 0) + Df(0, 0) \cdot (x, y) + \frac{1}{2}(x, y)Hf(0, 0)\begin{pmatrix} x \\ y \end{pmatrix} \]

Expanding out the linear algebra we obtain

\[ T_2(x, y) = 1 + xy \]

Which is the degree 2 Taylor polynomial about \((0, 0)\) of the function \(f(x, y) = e^{xy}\)

2. Find and classify all critical points of the function

\[ f(x, y) = x^3 + x^2 y + y^2 + xy + x + 1 \]

As in the last problem we will begin by computing both the matrix of partial derivatives and the Hessian matrix.

\[ Df(x, y) = (3x^2 + 2xy + y, x^2 + 2y + x) \]
\[ Hf(x, y) = \begin{pmatrix} 6x + 2y & 2x + 1 \\ 2x + 1 & 2 \end{pmatrix} \]

Now to find the critical values we compute where \(Df(x, y) = (0, 0)\)

We easily find the following points \((0, 0)\), \((1, -1)\), and \((1/2, -3/8)\). Now using our criteria on each of these points we find that the points \((0, 0)\) and \((1, -1)\) are both saddle points, while \((1/2, -3/8)\) is a local minimum.
3. Compute the matrix of partial derivatives of $f \circ g$ at the point $(0, 0)$ where

$$f(x, y) = (x^2 + y^2, x - y)$$

$$g(x, y) = (e^x - 3, 2y + 1)$$

This is a chain rule problem. We begin by computing $Df$ and $Dg$

$$Df = \begin{pmatrix} 2x & 2y \\ 1 & -1 \end{pmatrix}$$

$$Dg = \begin{pmatrix} e^x & 0 \\ 0 & 2 \end{pmatrix}$$

We then recall where we must evaluate the matrix of partial derivatives. As we are computing $f \circ g$ we evaluate

$$g(0, 0) = (-2, 1)$$

$$Dg(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

We then compute $Df(g(0, 0))$

$$Df(-2, 1) = \begin{pmatrix} -4 & 1 \\ 1 & -1 \end{pmatrix}$$

Now the chain rule tells us that $D(f \circ g) = Df(g(0, 0)) \cdot Dg(0, 0)$ which we can easily compute

$$\begin{pmatrix} -4 & 2 \\ 1 & -2 \end{pmatrix}$$

4. Using Green’s Theorem, compute the area of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

To use green’s theorem we will replace the double integral with a line integral over the function $F = (-y/2, x/2)$, so that we are finding area. We then must parameterize the boundary of the ellipse, which we can easily do by modifying the equations of circles.

$$c(t) = (5 \cos(t), 2 \sin(t)) \quad \text{for } 0 \leq t \leq 2\pi$$

We then can setup and compute the line integral
\[
\frac{1}{2} \int_0^{2\pi} (-2 \sin(t), 5 \cos(t)) \cdot (-5 \sin(t), 2 \cos(t)) dt = \frac{1}{2} \int_0^{2\pi} (10 \sin^2(t) + 10 \cos^2(t)) dt \\
= \frac{1}{2} \int_0^{2\pi} 10 dt \\
= 10\pi
\]

5. Using Stokes’s Theorem, compute the value of the line integral

\[
\oint_C \mathbf{F} \cdot d\mathbf{S}
\]

Where \( \mathbf{F}(x, y, z) = (\tan(x^2 + x, y - 2yz, \cos(z^4)) \) and \( C \) is the boundary of the region \( z^2 = x^2 + y^2 \) above \( z = 0 \) and below \( z = 1 \) (with upward facing normal vector).

We will first parameterize the surface \( S \) as

\[ s(r, \theta) = (2r \cos(\theta), 3r \sin(\theta), r) \quad \text{for } 0 \leq r \leq 1 \text{ and } 0 \leq \theta \leq 2\pi \]

We will also take a second and compute a normal vector to the surface, anticipating its use later.

\[
\text{Curl } \mathbf{F} = \left( \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{j} \\
2 \cos(\theta) & 3 \sin(\theta) & 1 \\
-2u \sin(\theta) & 3r \cos(\theta) & 0
\end{array} \right) = (-3r \cos(\theta), -2u \sin(\theta), 6r)
\]

Using Stokes’s theorem we can convert the integral over the boundary to an integral over the entire surface by changing the integrand into the curl. Let’s compute \( \text{Curl } \mathbf{F} \)

\[
\text{Curl } \mathbf{F} = \left( \begin{array}{ccc}
\mathbf{i} & \mathbf{j} & \mathbf{j} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\tan(x^2 + x, y - 2yz, \cos(z^4))
\end{array} \right) = (y^2, 0, 0)
\]

Then we can easily setup the surface integral over \( \text{Curl } \mathbf{F} \)

\[
\int_0^{2\pi} \int_0^1 (9r^2 \sin^2(\theta), 0, 0) \cdot (-3r \cos(\theta), -2u \sin(\theta), 6r) r \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (-27r^3 \sin^2(\theta) \cos(\theta)) r \, dr \, d\theta \\
= -27 \int_0^{2\pi} \int_0^1 \sin^2(\theta) \cos(\theta) r^4 \, dr \, d\theta \\
= -27 \int_0^{2\pi} \int_0^1 \sin^2(\theta) \cos(\theta) \, d\theta \\
= 0
\]
Which we is the value of the line integral.

6. Use the Divergence Theorem to compute the value of the flux integral

\[ \iint_S \mathbf{F} \cdot d\mathbf{S} \]

Where \( \mathbf{F}(x, y, z) = (y^3 + 3x, xz + y, z + x^4 \cos(x^2y)) \) and \( S \) is the boundary of the region bounded by \( x^2 + y^2 = 1, \ x \geq 0, \ y \geq 0 \) and \( 0 \leq z \leq 1 \)

Using the divergence theorem we will compute this rather difficult integral into a (hopefully) simpler triple integral over the divergence of \( \mathbf{F} \). To begin let us compute \( \nabla \cdot \mathbf{F} \)

\[ \nabla \cdot (\mathbf{F}) = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (y^3 + 3x, xz + y, z + x^4 \cos(x^2y)) = 5 \]

We then can setup the triple integral as

\[ \iiint_W 5 \, dV \]

Looking at the region we want to compute in cylindrical coordinates as

\[ \int_0^{\pi/2} \int_0^1 \int_0^1 5r \, dr \, d\theta \, dz = \frac{5\pi}{4} \]

Which is the value of the surface integral, by the Divergence theorem.