1. Suppose I give you the linear transformations

\[
R(x, y, z) = (x + z, 2x + 3y, y + 3z) \\
S(x, y) = (x + y, 2x - y) \\
T(x, y) = (x + y, 2x - 3y)
\]

(a) Write \( R \), \( S \) and \( T \) as matrices.

We would like to find matrices \( R \), \( S \) and \( T \) such that

\[
R \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x + z, 2x + 3y, y + 3z) \\
S \begin{bmatrix} x \\ y \end{bmatrix} = (x + y, 2x - y) \\
T \begin{bmatrix} x \\ y \end{bmatrix} = (x + y, 2x - 3y)
\]

By taking the coefficients of \( x \) and \( y \) in the function definitions, we find matrices:

\[
R = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} \\
S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} \\
T = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}
\]

And it is easy to check that these have the desired properties.

(b) Compute \( S \circ T \).

Since we have written these as matrices, we can compute composition by means of matrix multiplication. This gives

\[
S \circ T = ST = \begin{pmatrix} 3 & -2 \\ 2 & -3 \\ 0 & 5 \end{pmatrix}
\]

Or if we wanted to write this as an equation: \( S \circ T = (4x - 2y, 2x - 3y, 5y) \)

(c) Compute \( T \circ S \).

It is impossible to compute \( T \circ S \). The composition is not defined.
(d) Compute $R \circ S$

Since we have written these as matrices, we can compute composition by means of matrix multiplication. This gives

$$R \circ S = RS = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 2 & 5 \\ 6 & -2 \end{pmatrix}$$

(e) Compute $T^{-1}$.

In order to compute the inverse, we want to find a matrix $A^{-1}$ such that $A.A^{-1} = I$

The matrix inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So we know that

$$T^{-1} = \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & -1/5 \end{pmatrix}$$

(f) Is $R$ invertible?

We recall that a linear transformation $R$ is invertible as long as $R$ has a non-zero determinant. So we will compute the determinant of this linear transformation (written as a matrix).

$$\det \begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 1(3 \cdot 3 - 1 \cdot 0) - 0(2 \cdot 3 - 0 \cdot 0) + 1(2 \cdot 1 - 3 \cdot 0) = 11$$

Note that we have shown that $R$ is invertible without the need to compute the inverse.

2. Sketch the level curves of the function for $c = -2, -1, 0, 1, 2$

$$f(x, y) = x^2 - 2y^2 \quad f(x, y) = -x^2 - y^2$$

In order to compute level curves for the function, we look at the curves $c = x^2 - 2y^2$ for the given values of $c$, this gives 5 curves, given by

$$C_1 : -2 = x^2 - 2y^2$$
$$C_2 : -1 = x^2 - 2y^2$$
$$C_3 : 0 = x^2 - 2y^2$$
$$C_4 : 1 = x^2 - 2y^2$$
$$C_5 : 2 = x^2 - 2y^2$$
If we plot each of these equations, this is the picture we obtain.

For the curious, this is what the surface looks like

Likewise to compute the level curves for the function $f(x, y) = -x^2 - y^2$ will do exactly the same thing.

\[
\begin{align*}
C_1 &: -2 = -x^2 - 2y^2 \\
C_2 &: -1 = -x^2 - 2y^2 \\
C_3 &: 0 = -x^2 - 2y^2 \\
C_4 &: 1 = -x^2 - 2y^2 \\
C_5 &: 2 = -x^2 - 2y^2
\end{align*}
\]
The fact that there are only 3 curves in this picture should not bother anyone, for \( c = 1 \) and \( c = 2 \) this function has no real solutions.

For the curious, this is what the surface looks like

Identify each contour plot.

(a) \( e(x, y) = x^2 + y^2 \)

(b) \( f(x, y) = \sqrt{x^2 + y^2} \)

(c) \( g(x, y) = -x^2 - y^2 \)

(d) \( h(x, y) = 5x^2 + y^2 \)