1. Let \( f(x, y, z) = xy^3 \). Evaluate the line integral \( \int_C f(x, y)ds \) over the curve \( C(t) = (4\sin t, 4\cos t, 3t) \) where \( 0 \leq t \leq \pi/2 \). (If you are confused how to start, here are some steps to try)
   - What are the start and end points of the path \( C(t) \)?
   - Write out \( f(C(t)) \) so that \( f \) is a function of only \( t \)?
   - Compute \( C'(t) \)
   - Setup the line integral (is it a vector or scalar line integral?)
   - Evaluate the line integral

The first thing we need to notice is that this is a scalar valued function. Scalar valued function means scalar line integral. To begin we look at \( f(x, y) \) and replace every \( x \) with \( 4\sin t \) and every \( y \) with \( 4\cos t \). This gives the equation:

\[
 f(t) = 4\sin t64\cos^3 t
\]

The second thing we need is \( C'(t) \) Computing \( C'(t) \) is straightforward. We get \( C'(t) = (4\cos(t) - 4\sin(t), 3) \) Since we are dealing with the scalar line integral we mostly care about \(|C'(t)|\). The magnitude is

\[
 |C'(t)| = \sqrt{16\cos(t)^2 + 16\sin(t)^2 + 3^2} = 5
\]

We are now ready to setup and evaluate the integral. Setup is the easy part:

\[
 \int_C f(t) \cdot |C'(t)|dt = \int_0^{\pi/2} 4\sin t64\cos^3 t dt
\]

We will pull out the constants

\[
 = 1280 \int_0^{\pi/2} \sin t \cos^3 t dt
\]

Which leaves us with this integral which we evaluate to find our answer of 320. We easily integrate this with the substitution \( u = \cos t \)

\[
 1280 \int_0^{\pi/2} \sin t \cos^3 t = 1280 \int_0^{\pi/2} u^3 du = 4 \left. u^4 \right|_0^{\pi/2} = 320 \cos(t)^4 \left|_0^{\pi/2} = 320(1 - 0) = 320
\]

2. Let \( G(x, y) = (y, -x) \) Compute \( \oint_C G ds \) for a circle with radius 1 centered at the origin using the line integral. (Hint: use polar coordinates for your parameterization). (Again, here are some steps to follow if you don’t know where to start).
• sketch a graph of the vector field $G(x, y)$.
• Create a parametric equation for the curve $C$
• Compute $C'(t)$
• Write an out the equation of the vector field as you move along the curve.
• Evaluate the line integral

The first thing to do for this problem is to create a parametric curve for $C$. One choice would be

$$(\cos(t), \sin(t)) \text{ with } 0 \leq t \leq 2\pi$$

(As a side remark. Whenever anyone asks you to find a parametric equation for anything involving a circle, this should be the first equation to try. Also note that a parametric equation is not really complete without bounds on $t$) With this as our parametric equation we know

$$C'(t) = (-\sin t, \cos t)$$

The next step is to setup the integral we want to compute.

$$\int_c f(x, y) ds = \int_0^{2\pi} (\sin t, -\cos t).(-\sin t, \cos t) dt$$

And computing this integral we end up with

$$\int_c f(x, y) ds = -2\pi$$

3. Evaluate $\int_C (x^2 - y) dx + (x - y^2) dy$ where $C$ is the counterclockwise path around the region bounded by $y = x^2$ and $y = 6x + 7$

To begin setting up the contour integral we need to find parametric equations for the boundary. The boundary looks like this:

![Boundary Diagram]

We will need to break this integral up into two parts, a line segment, and a part of the parabola. To compute the entire integral we will take the sum of these two parts.

We can find the intersection points by setting the equations equal and solving. That will be the quadratic equation $x^2 - 6x - 7$ which will have roots of $-1$ and $7$
So we start parameterizing the curve, going counterclockwise we start at \((7, 49)\) and move to \((-1, 1)\) the parametric equation we get is

\[
(7 - 8t, 49 - 48t) \quad 0 \leq t \leq 1
\]

So with a parametric equation we can setup the vector line integral the derivative of the parametric equation will be \((-8, -48)\)

\[
\int_{0}^{1} \left( (7 - 8t)^2 - (49 - 48t), 7 - 8t - (49 - 48t)^2 \right) \cdot (-8, -48) \, dt = \frac{117472}{3}
\]

That's the first half of the work. Now we need parametric equations for the parabola. The easiest choice is the trivial parameterization.

\[
(t, t^2) \quad -1 \leq t \leq 7
\]

Which gives the vector line integral

\[
\int_{-1}^{7} \left( t^2 - t^4, t - t^4 \right) \cdot (1, 2t) \, dt = -\frac{116960}{3}
\]

The sum of these two integrals gives an answer of \(\frac{512}{3}\)

We will return to this problem later when we have Green's theorem at our disposal. The use of this theorem will make this integral much easier to compute.

4. Let \(C\) be the path given as \((t^2, t^2, 2t)\) with \(0 \leq t \leq 1\). Define \(f(x, y, z) = \frac{z}{x+z}\), and compute

\[
\int_{C} f \, ds
\]

We would like to compute the integral

\[
\int_{C} \frac{y + z}{x + z} \, ds
\]

In the course of computing this integral we will want to know \(c'(t)\). Lets compute that now

\[
c'(t) = (2t, 2t, 2)
\]

and as a result

\[
|c'(t)| = \sqrt{4t^2 + 4t^2 + 4} = 2\sqrt{2t^2 + 1}
\]

We setup this integral as a path integral, or a scalar valued line integral. The resulting integral is
\[
\int_0^1 2t^2 + 2t \cdot 2\sqrt{2t^2 + 1} dt = \int_0^1 t \sqrt{2t^2 + 1} dt + 1z dt
\]

Using the \( u \) substitution of
\[
u = 2t^2 + 1 \quad du = 4tdt
\]
we have the following integral
\[
\int \sqrt{u} du = \frac{2}{3} u^{3/2} = \frac{2}{3} (2t^2 + 1)^{3/2} \bigg|_0^1 = \frac{2}{3} \left( 1 - \frac{3}{2} \right) \approx 2.797
\]

5. Let \( F(x, y, z) = (x + y, y + z, x + z) \) be a vector valued function.

To give a better feel for the vector field, here is a plot of the vector field.

(a) Compute the gradient of this vector field.
The gradient is something we have worked with a great deal before, and this question is really just to illustrate how \( \text{grad} \), \( \text{div} \), and \( \text{curl} \) are related.
\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (1, 1, 1)
\]

(b) Compute the divergence of the vector field.
\[
\dot{\nabla} F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 1 = 3
\]

(c) Compute the curl of the vector field.
We compute the curl as the cross product
\[
\text{curl } F = \nabla \times \vec{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (F_1, F_2, F_3) = \text{det} \left( \begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{array} \right)
\]

In our specific problem we know that

\[
F_1(x, y, z) = (x + y) \\
F_2(x, y, z) = (y + z) \\
F_3(x, y, z) = (x + z)
\]

Which gives us the specific formula

\[
\text{curl } F = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \left( x + y, y + z, x + z \right) = (-1, -1, -1)
\]

Evaluating this determinant gives us the vector \((-1, -1, -1)\)