1. Find both first-order partial derivatives of

\[ f(x, y) = e^{3y} \sin(x) \quad g(x, y) = xy^2 \ln(3x) \]

2. Now find the four second order partial derivatives of \( f(x, y) \) and \( g(x, y) \).

We will do problems 1 and 2 at the same time.

First we will find the first-order partial derivative of \( f(x, y) \)

\[
\frac{\partial f}{\partial x} = e^{3y} \cos(x) \\
\frac{\partial f}{\partial y} = -e^{3y} \sin(x)
\]

And the second order partials of \( f(x, y) \) will be

\[
\frac{\partial^2 f}{\partial x^2} = -e^{3y} \sin(x) \\
\frac{\partial^2 f}{\partial x \partial y} = 3e^{3y} \cos(x) \\
\frac{\partial^2 f}{\partial x \partial y} = 3e^{3y} \cos(x) \\
\frac{\partial^2 f}{\partial y^2} = 9e^{3y} \sin(x)
\]

Now for \( g(x, y) \) the first order partial derivatives are

\[
\frac{\partial g}{\partial x} = y^2 + y^2 \ln(3x) \\
\frac{\partial g}{\partial y} = 2xy \ln(3x)
\]

And the second order partials of \( f(x, y) \) will be

\[
\frac{\partial^2 g}{\partial x^2} = \frac{y^2}{x} \\
\frac{\partial^2 g}{\partial x \partial y} = 2y + 2y \ln(3x) \\
\frac{\partial^2 g}{\partial y \partial y} = 2y + 2y \ln(3x) \\
\frac{\partial^2 g}{\partial y^2} = 3x \ln(3x)
\]
3. A function $f(x, y)$ is harmonic if it satisfies the Laplace equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{1}$$

Show that $f(x, y) = x^3 - 3xy^2$ is harmonic.

To show that this function is harmonic we start by taking the second order partial derivatives of $f(x, y)$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$
$$\frac{\partial^2 f}{\partial y^2} = -6x$$

So for this specific $f$ we know

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x - 6x = 0$$

Thus this function is harmonic.

4. The heat equation is: $\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$. Show that $u(x, t) = e^{-k^2 t} \sin(x)$ is a solution of the heat equation.

We begin by computing the equations used in the heat equation:

$$\frac{\partial u}{\partial t} = -e^{-k^2 t} \sin(x)$$
$$\frac{\partial^2 u}{\partial x^2} = -e^{-k^2 t} k^2 \sin(x)$$

5. An ant is trying to get out of the parabolic bowl $z = x^2 + 3y^2$. Suppose the ant is currently at the point $x = 2, y = -1, z = 7$. In which direction should the ant set out in order to climb out of the bowl fastest? Should it follow a straight line path from then on?

The gradient $\nabla f(x, y) = (2x, 6y)$ points in the direction of steepest increase at each point $(x, y)$. So for $x = 2, y = -1$ the ant should head in the direction of the vector $\nabla f(2, -1) = (4, -6)$. That is given by the unit vector $n = (2, -3)/\sqrt{13}$.

The ant should not continue to walk in a straight line since the direction of the gradient changes. Indeed at the point $(x, y) = (2 + 2t/\sqrt{13}, -1 - 3t/\sqrt{13})$, the direction of steepest increase is the unit vector parallel to $\nabla f(x, y) = (4 + 4t/\sqrt{13}, -6 - 18t/\sqrt{13})$, which is not the same as the direction $n$ of the straight line.
6. Let \( f(x, y) = x^2 + \frac{1}{2}y^2 - 2x \) Find a point on the graph \( z = f(x, y) \) where the tangent plane is horizontal.

We recall from in class, lab, textbooks, and folklore that we can write the normal vector at a point as

\[
\hat{n} = \left( -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)
\]

So for our purposes we will have

\[
\hat{n} = (-2x + 2, -y, 1)
\]

So all we need to find a point \((x, y)\) where \(\hat{n} = (0, 0, 1)\). The choice of \((1, 0)\) is one such point. \(f(1, 0) = -1\) meaning that a point where the tangent plane is horizontal will be \((1, 0, -1)\).

Another good way to approach this part of the problem was to say “If the plane is horizontal then \(\frac{\partial f}{\partial x} = 0\) and \(\frac{\partial f}{\partial y} = 0\). This gives the same solution.”