1. Decide whether or not the following sets of vectors are linearly independent. If they are find the area/volume of the region formed by the vectors.

\[ \vec{v}_1 = (1, 2, 3) \]
\[ \vec{v}_2 = (-1, -2, -3) \]

This pair of vectors in \( \mathbb{R}^3 \) are linearly dependent, since \( v_1 = -v_2 \)

\[ \vec{u}_1 = (1, 3, 2) \]
\[ \vec{u}_2 = (1, 6, 4) \]

This pair of vectors is linearly independent. To prove this we compute \( v_1 \times v_2 \) as

\[ v_1 \times v_2 = \det \begin{pmatrix} i & j & k \\ 1 & 3 & 2 \\ 1 & 6 & 4 \end{pmatrix} = \]
\[ = \hat{i}(3*4 - 6*2) - \hat{j}(1*4 - 1*2) + \hat{k}(1*6 - 1*3) = \]
\[ = (0, -2, 3) \]

Which is non zero. Further we know the magnitude of the cross product is the area of the parallelogram. So the area is

\[ |(0, -2, 3)| = \sqrt{4 + 9} = \sqrt{13} \]

\[ \vec{w}_1 = (1, 4, 2) \]
\[ \vec{w}_2 = (-1, 5, 3) \]
\[ \vec{w}_3 = (2, 4, 6) \]

We know that the vectors are linearly independent since

\[ \det \begin{pmatrix} 1 & 4 & 2 \\ -1 & 5 & 3 \\ 2 & 4 & 6 \end{pmatrix} = 38 \neq 0 \]
The area of the parallelepiped is then computed by the equation \((v_1 \times v_2).v_3 = 38\). It is a general fact the determinant of 3 vectors is equal to the the area of the parallelepiped.

\[
\vec{x}_1 = (1, 3, 2) \\
\vec{x}_2 = (1, 6, 4) \\
\vec{x}_3 = (3, 12, 8)
\]

In this case we know the set of vectors is linearly dependent, since

\[
\det \begin{pmatrix}
1 & 3 & 2 \\
1 & 6 & 4 \\
3 & 12 & 8
\end{pmatrix} = 0
\]

2. Let a triangle \(T\) be have edges \(u_1, u_2\) (as defined above) and \(u_1 - u_2\). Find the area of \(T\).

Thinking geometrically we realize that the area of \(T\) is half the area of the parallelogram spanned by \(u_1\) and \(u_2\). So the area of \(T\) is \(\sqrt{3}/2\).

3. Give parametric equations for a plane containing the lines

\[
\ell_1 = (t + 2, 3t - 5, 5t + 1) \\
\ell_2 = (5 - t, 3t - 10, 9 - 2t)
\]

The first step is to find two linearly independent vectors in the plane. Reversing our process for finding parameterizations for lines I pick the vectors \(v_1 = (1, 3, 5)\) and \(v_2 = (-1, 3, -2)\). It is easy to see that these are linearly independent. We also need a point in the plane. Any point will work so I choose the point \(t = 0\) on the line \(\ell_1\), which is \((2, -5, 1)\). This gives us a parametrization of the plane.

\[
x(r, s) = r - s + 2 \\
y(r, s) = 3(r + s) - 5 \\
z(r, s) = 5r - 2s + 1
\]
Next I would like to give the plane as a normal vector and a point. I take \(v_1\) and \(v_2\) and compute their cross product to find a normal vector.

\[
v_1 \times v_2 = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 5 \\ -1 & 3 & -2 \end{pmatrix} = -21\hat{i} - 3\hat{j} + 6 = (-21, 3, 6)
\]

So the same plane is specified by the normal vector \((-21, 3, 6)\) and the point \((2, -5, 1)\)

This gives rise to the explicit equation of the plane. Using the equation \(n.(x-a) = 0\) we have an equation for the plane of

\[
(-21, -3, 6). (x - 2, y + 5, z - 1) = 21 - 21x - 3y + 6z = 0
\]

To find the equation for the plane orthogonal to the line \(\ell_1\) passing through the point \(\ell_2(0)\). We begin by finding a vector on \(\ell_1\), we compute

\[
\ell_1(0) - \ell_1(1) = (1, 3, 5)
\]

We will treat this vector as our normal vector. To find the equation of a plane we also need a point in the plane, in this case \(\ell_2(0)\). We can easily compute this point

\[
\ell_2(0) = (5, -10, 9)
\]

And using the equation to define a plane from a normal vector, we have

\[
(1, 3, 5). (x - 5, y + 10, z - 9) = 0
\]

Which we will expand out to find

\[
x - 5 + 3y + 30 + 5z - 45 = 0
\]

\[
x + 3y + 5z = 20
\]

Which is an equation for the desired plane.
4. A plane is given by the equation

\[ 3x + 7y - z = 0 \]

- Find parametric equations for the same plane

The first thing we want to do in order to find parametric equations for the plane, is find 3 points laying in the plane. To do this we can just pick three pairs of number \((x, y)\) and solve for the \(z\) in the plane. Three easy choices are:

- \(x = 0, y = 0\)
- forcing \(z = 0\)
- \(P_1 = (0, 0, 0)\)

- \(x = 1, y = 0\)
- forcing \(z = 3\)
- \(P_2 = (1, 0, 3)\)

- \(x = 0, y = 1\)
- forcing \(z = 7\)
- \(P_3 = (0, 1, 7)\)

We can then find displacement vectors

\[ v_1 = \vec{P}_1 P_2 = (1, 0, 3) \]
\[ v_2 = \vec{P}_1 P_3 = (0, 1, 7) \]

So using the point \((0, 0, 0)\), we can write a nice parametric equation for the plane as:

- Write the plane in the form of normal vector at a given point.

We can also take the two vectors that we found and compute a normal vector as their cross product.

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & 3 \\
0 & 1 & 7 \\
\end{vmatrix} = (-3, -7, 1)
\]
So the normal vector is \((-3, -7, 1)\) which we combine with the point \((0, 0, 0)\) in order specify a plane.

- Find the intersection of the 2 planes given in problems 3 and 4.

It is easiest to compute the intersection of two planes from their explicit equations (rather than parametric or normal). In problem 3 we found the equation of the plane to be \(21 - 21x - 3y - z = 0\). We would like to find the intersection of the planes

\[
\begin{align*}
3x + 7y - z &= 0 \\
21 - 21x - 3y + 6z &= 0
\end{align*}
\]

If we multiply the top equation by \(-7\) we get the equivalent set of equations (which we will find more useful).

\[
\begin{align*}
-21x - 49y + 7z &= 0 \\
21 - 21x - 3y + 6z &= 0
\end{align*}
\]

a point which is in the intersection is one which is satisfied by both equations. We can thus set the equations equal and simplify

\[
\begin{align*}
-21x - 49y + 7z &= 21 - 21x - 3y + 6z \\
-49y + 7z &= 21 - 3y + 6z \\
(-49 + 3)y &= 21 + (6 - 7)z \\
46y + 21 &= z
\end{align*}
\]

This by itself is not a line in \(\mathbb{R}^3\), but we can use points in this plane to find points on the intersection. By using simple \(y\) values of 0 and 1 we get the points

\[
P_1 = (7, 0, 21) \\
P_2 = (20, 1, 67)
\]

We can verify that these points are on the line of intersection, by using the equations of the plane.
\[3x + 7y - z = 3 \cdot 7 + 7 \cdot 0 - 21 = 0\]
\[21 - 21x - 3y + 6z = 21 - 21 \cdot 7 - 3 \cdot 0 + 6 \cdot 21 = 0\]
\[3x + 7y - z = 3 \cdot 20 + 7 \cdot 1 - 67 = 0\]
\[21 - 21x - 3y + 6z = 21 - 21 \cdot 20 - 3 \cdot 1 + 6 \cdot 67 = 0\]

This in turn gives us a vector parallel to the line of intersection.

\[v = (13, 1, 46)\]

From a vector and a point we can parameterize a line in \(\mathbb{R}^3\). Reusing the point \((7, 0, 21)\) we obtain the parameterization

\[p(t) = (13, 1, 46)t + (7, 0, 21) = (13t + 7, t, 46t + 21)\]

And we can easily check that this is the line of intersection again by using the equations for the plane

\[3x + 7y - z = 3 \cdot (13t + 7) + 7 \cdot (46t) - 46t - 21 = 0\]
\[21 - 21x - 3y + 6z = 21 - 21(13 - t) - 3(46t) + 6(26t + 21) = 0\]

So we have a parametric equation of the line of intersection. There are many others.