1. For each of the following maps, decide if it is one-to-one and/or onto.

   (a) \( f(x, y) = (x + 2y, 2x + 4y) \) mapping \( \mathbb{R}^2 \to \mathbb{R}^2 \)

   This map is neither one-to-one nor onto. To see that is is not one-to-one we notice

   \[
   f(1, 1) = (3, 6) = f(2, 1/2)
   \]

   To see that this map is not onto we can write it as a matrix

   \[
   T(x, y) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}
   \]

   then we note that the matrix

   \[
   T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}
   \]

   has \( \det(T) = 0 \), so \( T \) has rank strictly less than 2. Therefore \( T \) is not onto.

   (b) \( g(x, y) = (2x + 4y, 3y) \) mapping \( \mathbb{R}^2 \to \mathbb{R}^2 \)

   This map is both one-to-one and onto.

   (c) \( h(x, y) = (x^2, y^4) \) mapping \( \mathbb{R}^2 \to [0, \infty) \times [0, \infty) \)

   This map is onto (for obvious reasons), but is not one-to-one since

   \[
   h(1, 1) = (1, 1) = h(-1, 1)
   \]

2. Evaluate the integral

   \[
   \int_0^1 \int_0^\sqrt{1-y^2} \int_0^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy
   \]

   We will begin by plotting the region of integration. You needn't do this if you understand the geometry, however, a picture will make things more clear. Here is what the region looks like, as plotted by mathematica.
This is somewhat difficult to see, so here is a cross section.

Although these pictures aren’t bad, they are not a particularly precise description. A better way of describing the region would be in equations. This region is the volume between \( z = x^2 + y^2 \) and \( z^2 = x^2 + y^2 \) bounded by the planes \( y = 0, x = 1 \) and \( z = 1 \).

It is often difficult to know what coordinate system to use, the two obvious choices here cylindrical and spherical. In general if a region has “messy” \( z \)-coordinates, its a good bet to try cylindrical coordinates.

To describe the region in cylindrical coordinates we say this is the volume between \( z = x^2 + y^2 \) and \( z^2 = x^2 + y^2 \) bounded by \( 0 \leq r \leq 1 \) and \( 0 \leq \theta \leq \pi/2 \).

3. Setup and evaluate the integral.

We then setup the integral as

\[
\int_0^1 \int_0^{\pi/2} \int_0^1 r \sqrt{x^2 + y^2} \, r \, dz \, dr \, d\theta
\]

If you work this integral all the way through you end up with an answer of

\[
\frac{1}{96}
\]

4. Compute the integral

\[
\int_D \sqrt{y^2/x} \, dA
\]
over the region $D$ defined by $0 \leq x^2/y \leq 1$ and $0 \leq y^2/x \leq 8$. (Hint: use the change of variables $u = \sqrt{y^2/x}$ and $v = \sqrt{x^2/y}$ and its inverse transformation $x = uv^2$ and $y = u^2v$.)

We will use the change of variables theorem to compute this integral. To begin, let's write down what we know in terms of the change of variables theorem.

We have $D$ being the region $0 \leq x^2/y \leq 1$ and $0 \leq y^2/x \leq 8$ which are not very useful bounds. We could try to solve for $x$ and $y$ and find bounds for integration, but it will be difficult, and give us a difficult integral.

Now let us consider what the region $D^*$ is. It is easy to setup bounds for $D^*$ since we have $0 \leq u^3 \leq 1$ and $v \leq v^3 \leq 8$ from the problem statement. This gives $0 \leq u \leq 1$ and $v \leq v^3 \leq 2$

The map $T : D^* \to D$ is then $T(u, v) = (uv^2, u^2v)$, so we must compute the Jacobian

$$\begin{vmatrix}
\frac{\partial(x, y)}{\partial u, v} \\
\frac{\partial(x, y)}{\partial u, v}
\end{vmatrix} = \det \begin{pmatrix} \frac{\partial uv^2}{\partial u} & \frac{\partial uv^2}{\partial v} \\
\frac{\partial u^2v}{\partial u} & \frac{\partial u^2v}{\partial v}
\end{pmatrix} = 3u^2v^2$$

So, we now setup the integral using the change of variables theorem.

$$\int \int_{D^*} 3u^3v^2 dA = 3 \int_0^1 \int_0^2 u^3v^2 dv du$$

Which we can evaluate to 2.

5. Find the volume of the region between a sphere with radius 1 and a sphere with radius 2, in the region of $\mathbb{R}^3$ bounded by $x \geq 0, y \geq 0, z \geq 0$

This was meant to be a fairly simple problem, you should get practice reading a word problem, and thinking in spherical coordinates from this problem.

The bounds $x \geq 0, y \geq 0, z \geq 0$ tell us that the region is entirely located in the section of the space with $x, y, z$ all non-negative. So the region should bet the “shell” of a sphere.

It will look like this
To parameterize this region we first note that $\rho$ can be anything between 1 and 2. Since this is all in the positive section of $\mathbb{R}^3$ we know that the most $\theta$ can be is $\frac{\pi}{2}$ and we likewise know that $\phi$ can be at most $\frac{\pi}{2}$.

This gives the following bounds on our variables

\[ 0 \leq \theta \leq \frac{\pi}{2} \]
\[ 0 \leq \phi \leq \frac{\pi}{2} \]
\[ 1 \leq \rho \leq 2 \]

The integral is then

\[ \int_{1}^{2} \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \rho^2 \sin(\phi) \, d\phi \, d\theta \, d\rho \]

6. Evaluate the integral over the region $R$ which is bounded by the lines $y = x, y = x - 2, y = -2x, y = 3 - 2x$

\[ \iint_{R} (3x + 4y) \, dA \]

Use the linear transformation $x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(v - 2u)$ to evaluate the integral.

For the sake of convincing you this isn’t an integral we’d want to do by hand, here is a picture of the region we are integrating over.

You will note that in order to do this integral without changing coordinates, we would need to setup 3 different integrals, each with different bounds. Yech!

We can however take from this picture the coordinates of the “corners” of the parallelogram. They will be

\[ (0, 0), (1, 1), \left( \frac{2}{3}, \frac{-4}{3} \right), \left( \frac{5}{3}, \frac{-1}{3} \right) \]

These coordinates are in the $x - y$ plane and we want to find what the preimage of the map is in the $u - v$ plane. You could solve these equations for $u$ and $v$ explicitly, but I will use some linear algebra in order to do it.
We know that for a point \((u, v)\) in the \(u - v\) plane that:

\[
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]

Lets name this matrix

\[
M = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix}
\]

Expanding this equation out gives us exactly the transformation we are given in the problem statement, and gives us points in the \(x - y\) plane. But we currently have points in the \(x - y\) plane so we want to invert this mapping. We do this by taking the inverse matrix.

\[
\begin{pmatrix}a & b \\
c & d\end{pmatrix}^{-1} = \frac{1}{(ad-bc)}\begin{pmatrix}d & -b \\
-c & a\end{pmatrix}
\]

In our specific case this gives our inverse being:

\[
A = \begin{pmatrix}1 & -1 \\
2 & 1\end{pmatrix}
\]

Now applying this matrix to the points we found earlier we get the new set of points.

\[
(0, 0)(0, 3)(2, 0)(2, 3)
\]

So we recognize this as a rectangle. We would like to now setup the integral but first we have to change the integrand.

\[
3x + 4y = \frac{1}{3}(-5u + 7v)
\]

which we do simply by using the given equations. This allows us to setup the integral (in the \(u, v\) coordinates)

\[
\int_{0}^{2} \int_{0}^{3} \frac{1}{3} (-5u + 7v) \ detM \ dudv = \frac{11}{3}
\]

Which is the same answer we would have gotten had we just blindly integrated. In this example it seems like we are doing a lot more work, but if you are comfortable with these sorts of change of variable problems, you can often save a lot of work this way. Its worthwhile going and looking through the book and making sure you see how I've used each of theorems, and how the regions are related.
7. Compute the integral

\[ \int \int \int_W \sqrt{x^2 + y^2} dV \]

Where \( W \) is the region inside the cylinder \( x^2 + y^2 = 1 \) with \( x \geq 0, y \geq 0, z \geq 0 \) and below the surface \( z = 2xy \)

This is a very typical cylindrical coordinates problem. To begin we will describe this region in cylindrical coordinates. This region is easily described by

\[ 0 \leq \theta \leq \frac{\pi}{2} \]
\[ 0 \leq r \leq 1 \]
\[ 0 \leq z \leq 2 \cos(\theta) \sin(\theta) = \sin(2\theta) \]

This allows us to setup the integral, recall that our Jacobian is simply \( r \).

\[ \int \int \int_0^1 r^2 \cdot rdzd\theta dr \]

This is now a fairly simple integral. We can evaluate this integral to be \( \frac{\pi}{10} \).