1. Find both first-order partial derivatives of
\[ f(x, y) = e^{3y} \sin(x) \quad \quad g(x, y) = xy^2 \ln(3x) \]

2. Now find the four second order partial derivatives of \( f(x, y) \) and \( g(x, y) \).

This should be a computation which is familiar from previous calculus classes. We find the first-order partial derivative of \( f(x, y) \) using our standard differentiation tricks. Keep in mind that when computing \( \frac{\partial}{\partial x} \) we view \( y \) as a constant.

\[
\frac{\partial f}{\partial x} = e^{3y} \cos(x) \\
\frac{\partial f}{\partial y} = 3e^{3y} \sin(x)
\]

Now for \( g(x, y) \) the first order partial derivatives are

\[
\frac{\partial g}{\partial x} = y^2 + y^2 \ln(3x) \\
\frac{\partial g}{\partial y} = 2xy \ln(3x)
\]

3. Find the matrix of partial derivatives of the function
\[ F(x, y, z) = (ze^{x^2+y^2} + xy, \cos(x^3y^2z^4)) \]

We see that \( F : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) so we know that we will have a \( 2 \times 3 \) matrix, and that we will need to compute 6 partial derivatives.

To begin we will rewrite

\[ F(x, y, z) = (f_1(x, y, z), f_2(x, y, z)) \]

Where

\[
f_1(x, y, z) = ze^{x^2+y^2} + xy \\
f_2(x, y, z) = \cos(x^3y^2z^4)
\]
So we will compute all the partial derivatives independently

\[
\begin{align*}
\frac{\partial f_1}{\partial x} &= y + 2xze^{x^2+y^2} \\
\frac{\partial f_1}{\partial y} &= x + 2yze^{x^2+y^2} \\
\frac{\partial f_1}{\partial z} &= e^{x^2+y^2} \\
\frac{\partial f_2}{\partial x} &= -3x^2y^2z^4 \sin(x^3y^2z^4) \\
\frac{\partial f_2}{\partial y} &= -2x^3yz^4 \sin(x^3y^2z^4) \\
\frac{\partial f_2}{\partial z} &= -4x^3y^2z^3 \sin(x^3y^2z^4)
\end{align*}
\]

\[DF = \begin{pmatrix} y + 2xze^{x^2+y^2} & x + 2yze^{x^2+y^2} & e^{x^2+y^2} \\
-3x^2y^2z^4 \sin(x^3y^2z^4) & -2x^3yz^4 \sin(x^3y^2z^4) & -4x^3y^2z^3 \sin(x^3y^2z^4) \end{pmatrix}\]

We then put these partial derivatives into the matrix of partial derivatives as follows

4. A function \( f(x, y) \) is harmonic if it satisfies the Laplace equation:

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{1}
\]

Show that \( f(x, y) = x^3 - 3xy^2 \) is harmonic.

To show that this function is harmonic we start by taking the second order partial derivatives of \( f(x, y) \)

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 6x \\
\frac{\partial^2 f}{\partial y^2} &= -6x
\end{align*}
\]

So for this specific \( f \) we know

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 6x - 6x = 0
\]

Thus this function is harmonic.
5. The heat equation is: \( \frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \). Show that \( u(x, t) = e^{-k^2 t} \sin(x) \) is a solution of the heat equation.

We begin by computing the equations used in the heat equation:

\[
\frac{\partial u}{\partial t} = -e^{-k^2 t} \sin(x)
\]
\[
\frac{\partial^2 u}{\partial x^2} = -e^{-k^2 t} k^2 \sin(x)
\]

6. Let \( f(x, y) = x^2 + \frac{1}{2} y^2 - 2x \). Find a point on the graph \( z = f(x, y) \) where the tangent plane is horizontal.

We recall from in class, lab, textbooks, and folklore that we can write the normal vector at a point as

\[
\hat{n} = \left( -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)
\]

So for our purposes we will have

\[
\hat{n} = (-2x + 2, -y, 1)
\]

So all we need to find a point \((x, y)\) where \(\hat{n} = (0, 0, 1)\). The choice of \((1, 0)\) is one such point. \( f(1, 0) = -1 \) meaning that a point where the tangent plane is horizontal will be \((1, 0, -1)\).

Another good way to approach this part of the problem was to say “If the plane is horizontal then \( \frac{\partial f}{\partial x} = 0 \) and \( \frac{\partial f}{\partial x} = 0 \). This gives the same solution.

7. Let \( f(x, y) = x/y + y/x \). Using a linear approximation about the point \((1/2, 1/4)\), estimate the value of \( f(0.48, 3) \).

We begin by recalling that the tangent plane gives a linear approximation to \( f(x, y) \) at a point, so we first compute the matrix of partial derivatives.

\[
Df(x, y) = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{y} - \frac{y}{x^2}, \frac{1}{x} - \frac{x}{y^2}
\end{pmatrix}
\]

Evaluating this at the point \((1/2, 1/4)\) we have

\[
Df(1/2, 1/4) = \begin{pmatrix}
\frac{1}{1/4} - \frac{1/4}{(1/2)^2}, \frac{1}{1/2} - \frac{1/2}{(1/4)^2}
\end{pmatrix}
= (3, -6)
\]
Now recall the form of the tangent plane \( L(x, y) \) at the point \((a, b)\).

\[
L(x, y) = f(a, b) + Df(a, b) \cdot (x - a, y - b)
\]

and for the point \((1/2, 1/4)\) this becomes

\[
L(x, y) = f(1/2, 1/4) + Df(1/2, 1/4) \cdot (x - 1/2, y - 1/4) \\
= \frac{5}{2} + (3, -6) \cdot (x - 1/2, y - 1/4) \\
= \frac{5}{2} + 3x - \frac{3}{2} - 6y + 6 \\
= \frac{5}{2} + 3x - 6y
\]

Now to estimate the value of \( f(.48, .3) \) we compute

\[
L(.48, .3) = \frac{5}{2} + 3 \cdot (.48) - 6 \cdot (.3) = 2.14
\]

Using a computer we can compute the exact value to be 2.225, and we see that our estimate is fairly close.

8. An ant is trying to get out of the parabolic bowl \( z = x^2 + 3y^2 \). Suppose the ant is currently at the point \( x = 2, y = -1, z = 7 \). In which direction should the ant set out in order to climb out of the bowl fastest? Should it follow a straight line path from then on?

The gradient \( \nabla f(x, y) = (2x, 6y) \) points in the direction of steepest increase at each point \((x, y)\). So for \( x = 2, y = -1 \) the ant should head in the direction of the vector \( \nabla f(2, -1) = (4, -6) \). That is given by the unit vector \( n = (2, -3)/\sqrt{13} \).

The ant should not continue to walk in a straight line since the direction of the gradient changes. Indeed at the point \((x, y) = (2 + 2t/\sqrt{13}, -1 - 3t/sqrt{13})\), the direction of of steepest increase is the unit vector parallel to \( \nabla f(x, y) = (4 + 4t/\sqrt{13}, -6 - 18t/\sqrt{13}) \), which is not the same as the direction \( n \) of the straight line.