1. Let \( g(x, y) = 2x^2 \cos(2\pi y^2) + y^2 \). You can use the fact that \( \mathbf{D}g(2, 1) = \begin{bmatrix} 8 & 2 \end{bmatrix} \).

(a) Find an equation for the tangent plane to the graph \( z = g(x, y) \) at the point \( (2, 1) \).

There are two key parts to this problem. The first, and easiest part was to evaluate the function at the point \( (2, 1) \). The computation was

\[
g(2, 1) = 2 \cdot 2^2 \cos(2\pi) + 1 = 8 + 1 = 9
\]

Once you have evaluated the function at \( (2, 1) \) we have to recall the equation for a tangent plane. This equation was

\[
z = L(x, y) = g(2, 1) + \mathbf{D}g(2, 1)(x - 2, y - 1)
\]

\[
= 8x + 2y - 9
\]

(b) Use the linear approximation of \( g(x, y) \) at the point \( (2, 1) \) to find an approximate value for \( g(1.9, 1.1) \).

To compute the linear approximation we just evaluate \( L(1.9, 1.1) = 8.4 \). In this case it is not a very good approximation as \( f(1.9, 1.1) \approx 3.005 \), however it is still the best linear approximation.

2. Consider the function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) defined by

\[
f(x, y) = \begin{cases} 
3x^2y + 2x^3 + 7y^3 \sqrt{x^4 + y^4} & \text{for } (x, y) \neq (0, 0), \\
0 & \text{for } (x, y) = (0, 0).
\end{cases}
\]

Use the limit definition to calculate the partial derivative \( \frac{\partial f}{\partial y}(0, 0) \).

By definition

\[
\frac{\partial f}{\partial x}(0, 0) = \lim_{k \to 0} \frac{f(0, 0 + k) - f(0, 0)}{k}
\]

\[
= \lim_{k \to 0} \frac{3 \cdot 0^2 \cdot k + 2 \cdot 0^3 + 7 \cdot k^3}{\sqrt{0^4 + k^4}} - 0
\]

\[
= \lim_{k \to 0} \frac{7k^3}{k^2}
\]

\[
= \lim_{k \to 0} 7
\]

\[
= 7
\]
3. Let \( g : \mathbb{R}^2 \to \mathbb{R}^3 \) be given by \( g(u, v) = (uv^2, u^2 + v^2, 2u - v) \). Find \( Dg(1, 1) \).

We know that \( Dg(u, v) \) will be a matrix, which we can easily compute to be

\[
Dg(u, v) = \begin{bmatrix}
v^2 & 2uv \\
u^2 & 2v \\
2 & -1
\end{bmatrix}
\]

So, we can evaluate this matrix at the point \((1, 1)\) in order to find the matrix \( Dg(1, 1) \).

\[
Dg(1, 1) = \begin{bmatrix}
1 & 2 \\
2 & 2 \\
2 & -1
\end{bmatrix}
\]

4. Let \( g \) be defined as in question 3. Let \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) be given by \( f(x, y, z) = (\sqrt{x^2 + 3} + \sqrt{z^2 + 3}, yz) \). Let \( h = f \circ g \). Using the Chain Rule, calculate \( Dh(1, 1) \).

First we compute \( Df(x, y, z) = \begin{bmatrix} x/\sqrt{x^2+3} & 0 & z/\sqrt{z^2+3} \\ 0 & 1 & y \end{bmatrix} \)

Now according to the chain rule we must compute \( Df(g(1, 1)) \), we know that \( g(1, 1) = (1, 2, 1) \) so evaluating we find

\[
Df(g(1, 1)) = \begin{bmatrix}
1/2 & 0 & 1/2 \\
0 & 1 & 2
\end{bmatrix}
\]

Now we will use this, together with the result of the previous problem to find

\[
Dh(1, 1) = Df(g(1, 1)) \cdot Dg(1, 1)
= \begin{bmatrix}
1/2 & 0 & 1/2 \\
0 & 1 & 2
\end{bmatrix} \cdot \begin{bmatrix}
1 & 2 \\
2 & 2 \\
2 & -1
\end{bmatrix}
= \begin{bmatrix}
3/2 & 1/2 \\
6 & 0
\end{bmatrix}
\]
5. Compute the value of \( x^2 \frac{\partial w}{\partial x} + y^2 \frac{\partial w}{\partial y} \) at the point \((1, -1)\) if \( w = f(\frac{x+y}{xy}) \) and \( f'(0) = -2 \).

Define
\[
u(x, y) = \frac{x + y}{xy} \]

Then \( f(\frac{x+y}{xy}) = f(u(x, y)) \) is defined in terms of function composition, so we should first think of the chain rule. The chain rule says
\[
Dw = Dw(x, y) = \left( \begin{array}{c} \frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial y} \end{array} \right) = D(f \circ u) = \nabla f(u(x, y)) \cdot Du(x, y)
\]

We can compute
\[
Du(x, y) = \left( \begin{array}{c} \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \end{array} \right) = \left( \begin{array}{c} -1 \\
-1 \end{array} \right)
\]

This allows us to compute \( \frac{\partial w}{\partial x} \) and \( \frac{\partial w}{\partial y} \) at the point \((1, -1)\)
\[
Dw(1, -1) = Df(0) \cdot (-1, -1) = (2, 2)
\]

Which finally allows us to compute
\[
1 \cdot 2 + 1 \cdot 2 = 4
\]

6. The temperature \( T(x, y) \) at a point \((x, y)\) in the plane is a differentiable function such that \( \frac{\partial T}{\partial x} (3, -1) = 2 \) and \( \frac{\partial T}{\partial y} (3, -1) = 5 \).

(a) Find the directional derivative of \( T \) at the point \((3, -1)\) in the direction given by the unit vector \( \mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2}) \).

The directional derivative in direction \( \mathbf{u} \) is given by
\[
\nabla T \cdot \mathbf{u} = (2, 5) \cdot \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \frac{7}{\sqrt{2}}.
\]

(b) A bug crawls along a level curve of the temperature function \( T \) at speed one. What are the possible velocity vectors \( \mathbf{v} \) for the bug at the moment it crawls through the point \((3, -1)\)?
The directional derivative for \( T \) in the direction of a level curve must be zero. This gives the equation \( \nabla T \cdot v = 0 \). The speed of the bug is one, so we also have \( \|v\| = 1 \).

Then the equation \( \nabla T \cdot v = 0 \), can be written as \( 2v_1 + 5v_2 = 0 \). This gives \( v_2 = -(2/5) \, v_1 \). One can also note by inspection that \( (-5, 2) \) is a solution and that any multiple of \( (-5, 2) \) is a solution. Since \( \|v\| = 1 \), \( (-5, 2) / \sqrt{29} \) or \( (5, -2) / \sqrt{29} \), take note that we have carefully normalized the solution to unit length.

7. Find a direction for which the directional derivative of the function \( w(x, y, z) = y(x^2 + z^2) - z^3 \) at the point \((1, 1/2, 1)\) is zero.

In general we compute the directional derivative based on the gradient. We can compute the partial derivatives

\[
\begin{align*}
\frac{\partial w}{\partial x} &= 2xy \\
\frac{\partial w}{\partial y} &= x^2 + z^2 \\
\frac{\partial w}{\partial z} &= 2yz - 3z^2
\end{align*}
\]

Which gives the gradient, evaluated at the point \((1, 1/2, 1)\)

\[
\nabla f(1, 1/2, 1) = (2xy, x^2 + z^2, 2yz - 3z^2) = (1, 2, -2)
\]

The directional derivative is given as \( \nabla f \cdot u \) so we must find a vector \( u = (x, y, z) \) such that \( \nabla f \cdot u = 0 \). Expanding this we have

\[
0 = \nabla f \cdot u = x + 2y - 2z = 0
\]

Now we just need to find a solution to this equation. Setting \( x = 1 \) \( y = 1 \) we can solve \( z = 3/2 \). This gives the vector \((1, 1, 3/2)\) and we can check that it is in fact a direction gives 0 change in \( w \), since

\[
(1, 2, -2) \cdot (1, 1, 3/2) = 1 + 2 - 3 = 0
\]

8. Calculate the directional derivative of \( f(x, y, z) = x \cos(y) \sin(z) \) at the point \( a = (1, \pi/4, 5\pi/6) \) in the direction of \( u = (3, 0, 1) \)

The first thing to do with a problem like this is to convert \( \vec{v} \) into a unit vector. To find the unit vector \( \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \),

\[
\vec{v} = \frac{(3, 0, 1)}{\sqrt{3^2 + 1^2}} = \frac{(3, 0, 1)}{\sqrt{10}} = \left( \frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}} \right)
\]

\[
\vec{u} = \frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}
\]
Now we will proceed to compute the gradient at of $f$ at the point
\[ \nabla f(x, y, z) = (\cos(y) \sin(z), -x \sin(y) \sin(z), x \cos(y) \cos(z)) \]

Evaluating at the point \((1, \frac{\pi}{4}, \frac{5\pi}{6})\) we have
\[ \nabla f \left(1, \frac{\pi}{4}, \frac{5\pi}{6}\right) = \left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{4}\right) \]

Then using the formula for directional derivatives this is
\[ \left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{6}}{4}, -\frac{\sqrt{6}}{4}\right) \cdot \left(\frac{3}{\sqrt{10}}, 0, \frac{1}{\sqrt{10}}\right) = \frac{3 + \sqrt{3}}{4\sqrt{5}} \]

9. For the following regions $R$, compute the specified regions.

(a) $R$ is the rectangle with vertices \((-2, 2), (0, 2), (0, 0), (-2, 0)\)
\[ \int \int_R (3x^2 + 2y^3) dA \]

We would usually draw a picture of the region here, but this region is very simple, so there is no need. We can see from the points that $x$ varies freely between $-2$ and $0$, while $y$ varies freely between $0$ and $2$. This gives the setup for the integral
\[ \int_0^2 \int_{-2}^0 (3x^2 + 2y^3) \, dx \, dy \]

Which we can compute
\[ \int_0^2 \int_{-2}^0 (3x^2 + 2y^3) \, dx \, dy = \int_0^2 \left( x^3 + 2y^3 \right)_{-2}^0 \, dy \, dx = \]
\[ = \int_0^2 (8 + 4y^3) \, dy \, dx = \]
\[ = 8y + y^4 \bigg|_0^2 = 32 \]
(b) \( R \) is the region bounded by \( x = y^2 \) and \( x = 5y \)

\[
\int \int_{R} ye^x \, dA
\]

First we setup the bound on this integral. The easiest way to do this is to draw a picture

![Diagram of the region R bounded by \( x = y^2 \) and \( x = 5y \)]

We see that the bounds on \( y \) vary between between \( y = 0 \) and \( y = 5 \). Further we see that for any \( y \), \( x \) can vary between \( y^2 \) and \( 5x \). This gives us all the information we need to setup the integral.

\[
\int_{0}^{5} \int_{y^2}^{5y} ye^x \, dx \, dy
\]

Now we can compute this integral

\[
\int_{0}^{5} \int_{y^2}^{5y} ye^x \, dx \, dy = \int_{0}^{5} ye^x \bigg|_{y^2}^{5y} = \\
= \int_{0}^{5} ye^{5y} - ye^{y^2} = \\
= \left( \frac{ye^{5y}}{5} - e^{y^2} \right) \bigg|_{0}^{5} = \\
= \frac{27 + 23e^{25}}{50}
\]