1. Consider the integral: $\int_0^1 \int_y^1 x^2 \, dx \, dy$

(i) Compute the Integral.
Computing this integral as given is straightforward.

\[
\int_0^1 \int_y^1 x^2 \, dx \, dy = \int_0^1 \left[ \int_0^1 \frac{1}{3} x^3 \right]_y^1 dy \\
= \int_0^1 \frac{1}{3} - \frac{1}{3} y^3 \, dy \\
= \frac{1}{3} y - \frac{1}{12} y^4 \bigg|_0^1 \\
= \frac{1}{4}
\]

(ii) Sketch the region you are integrating over.

(iii) Change the order of integration.
Reversing the order of integration we look now want to setup the integral is taken $dy \, dx$. So we look along the $x$ axis and see that our integral will still go between 0 and 1. Then we look along the $y$ axis, and for any $x \in (0, 1)$ we know $0 \leq y \leq x$. This gives us all the information we need to setup this integral.

\[
\int_0^1 \int_0^x x^2 \, dy \, dx
\]

(iv) Compute the new integral.
This computation is even easier.
\[
\begin{align*}
\int_0^1 x^2 \, dy \, dx &= \int_0^1 x^2 y \bigg|_0^1 \\
&= \int_0^1 x^3 \\
&= \frac{1}{4} x^4 \bigg|_0^1 \\
&= \frac{1}{4}
\end{align*}
\]

The fact that the answers agree should come as no surprise.

2. Consider the integral: \(\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3 \, dy \, dx\)

(i) Compute the Integral.

This integral is very similar to the integrals we were computing in mathematica, only easier.

\[
\begin{align*}
\int_0^1 \int_0^{\sqrt{1-x^2}} 3 \, dx \, dy &= 3 \int_0^1 \sqrt{1-x^2} \\
&= 3 \int_0^1 \arcsin(x) \, dx \\
&= \sqrt{1-x^2} + x \arcsin(x) \bigg|_0^1 = \frac{3\pi}{4}
\end{align*}
\]

(ii) Sketch the region you are integrating over.
(iii) Change the order of integration.

This is something of a trick question. In this problem you can reverse the order of integration and have the integral

$$
\int_{0}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3dx\,dy
$$

(iv) Compute the new integral.

Since changing the order of integration did not change the integral there is no need to recompute the new integral.

3. Consider the integral: \( \int_{0}^{3} \int_{0}^{\sqrt{y}} (x + y)\,dx\,dy \)

(i) Compute the Integral.

$$
\int_{0}^{3} \int_{0}^{\sqrt{y}} (x + y)\,dx\,dy = \int_{0}^{3} \left[ \frac{x^2}{2} + xy \right]_{0}^{\sqrt{y}}
= \int_{0}^{3} \left( \frac{9}{2} + \frac{5y}{2} - y^{3/2} \right)\,dy
= \frac{1}{2} \left( 9y + \frac{5y^2}{2} - \frac{4y^{5/2}}{5} \right) \bigg|_{0}^{9} = \frac{891}{20}
$$

(ii) Sketch the region you are integrating over.

(iii) Change the order of integration.

$$
\int_{0}^{3} \int_{0}^{x^2} (x + y)\,dy\,dx
$$
(iv) Compute the new integral.

\[ \int_{0}^{3} \int_{0}^{\frac{x^2}{2}} (x + y) \, dy \, dx = \int_{0}^{3} \left( xy + \frac{x^2}{2} \right) \bigg|_{0}^{\frac{x^2}{2}} \, dx \]

\[ = \int_{0}^{3} x^3 + \frac{x^4}{2} \, dx \]

\[ = \left. \frac{1}{20} (x^4 (5 + 2x)) \right|_{0}^{3} = \frac{891}{20} \]

And although it should not really be a surprise, the two integrals are equal.

4. Setup and evaluate the integral over the region \( V \) bounded by a cube, centered at \((0, 1, 2)\) on which all edges have length 2.

\[ \iiint_{V} xyz \, dV \]

The correct way to setup this integral is

\[ \int_{0}^{2} \int_{1}^{3} \int_{-1}^{1} xyz \, dx \, dz \, dy = \int_{0}^{2} \int_{1}^{3} \left( \frac{y z^2}{2} \right)_{-1}^{1} \, dz \, dy = \int_{0}^{2} \int_{0}^{3} 0 \, dz \, dy = 0 \]

So we find the value of the integral to be zero with no further work.

5. Evaluate the triple integral.

\[ \int_{-1}^{2} \int_{0}^{3} \int_{0}^{\frac{y+z}{2}} 3yz^2 \, dx \, dy \, dz \]

6. Consider but do not evaluate the integral

\[ \int_{-2 \sqrt{4-x^2}}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} 2 \, dz \, dy \, dx \]

(a) What is the region of integration?

The region of integration is the section of the sphere of radius 2 given by \( x^2 + y^2 + z^2 = 4 \) with \( z \geq 0 \). In particular it is a hemisphere centered at \((0, 0, 0)\) located entirely above the \( xy \) plane.

(b) Can you find the value of this integral without integrating?

This integral we can compute without any knowledge of calculus. The value of the integral will be twice the volume of a hemisphere. The area of a hemisphere will be half the volume of a sphere.
The volume of a sphere of radius $r$ is given by

$$V = \frac{4}{3} \pi r^3$$

So the value of the integral will be

$$2 \cdot \frac{1}{2} \cdot \frac{4}{3} \pi \cdot \frac{8\pi}{3} = \frac{32\pi}{3}$$