1. Let \( f(x, y, z) = xy^3 \). Evaluate the line integral \( \int_C f(x, y)ds \) over the curve \( C(t) = (4 \sin t, 4 \cos t, 3t) \) where \( 0 \leq t \leq 1/2 \). Is this a line integral or a path integral?

   (a) Write out \( f(C(t)) \) so that \( f \) is a function of only \( t \)?

   (b) Compute \( C'(t) \)

   (c) Evaluate the line integral

   The first thing we need to notice is that this is a scalar valued function. Scalar valued function means scalar line integral. To begin we look at \( f(x, y) \) and replace every \( x \) with \( 4 \sin t \) and every \( y \) with \( 4 \cos t \). This gives the equation:

   \[
   f(t) = 4 \sin t 64 \cos^3 t
   \]

   The second thing we need is \( C'(t) \) Computing \( C'(t) \) is straightforward. We get \( C'(t) = (4 \cos t - 4 \sin t, 3) \) Since we are dealing with the scalar line integral we mostly care about \( |C'(t)| \). The magnitude is

   \[
   |C'(t)| = \sqrt{16 \cos(t)^2 + 16 \sin(t)^2 + 3^2} = 5
   \]

   We are now ready to setup and evaluate the integral. Setup is the easy part:

   \[
   \int_C f(t) \cdot |C'(t)| dt = \int_0^{\pi/2} 4 \sin t 64 \cos^3 5 dt
   \]

   We will pull out the constants

   \[
   = 1248 \int_0^{\pi/2} \sin t \cos^3 t
   \]

   Which leaves us with this integral which we evaluate it to find our answer of 320

2. Let \( g(x, y) = (y, -x) \) Compute \( \oint_C gds \) for a circle with radius 1 centered at the origin using the line integral. (Hint: use polar coordinates for your parameterization).

   (a) Write out \( f(C(t)) \) so that \( f \) is a function of only \( t \)?

   (b) Compute \( C'(t) \)

   (c) Evaluate the line integral

   The first thing to do for this problem is to create a parametric curve for \( C \). One choice would be

   \[
   (\cos(t), \sin(t)) \text{ with } 0 \leq t \leq 2\pi
   \]

   (As a side remark. Whenever anyone asks you to find a parametric equation for anything involving a circle, this should be the first equation to try. Also note that a parametric equation is not really complete without bounds on \( t \)) With this as our parametric equation we know
\[ C'(t) = (-\sin t, \cos t) \]

The next step is to setup the integral we want to compute.

\[ \int_C f(x, y)ds = \int_0^{2\pi} (\sin t, -\cos t).(-\sin t, \cos t)dt \]

And computing this integral we end up with

\[ \int_C f(x, y)ds = -2\pi \]

3. Consider the integral \( \int_C \mathbf{F} \cdot d\mathbf{s} \) with \( \mathbf{F}(x, y) = F_1(x, y), F_2(x, y) \) continuous functions

(a) What conditions on the curve \( C \) and/or the vector field \( \mathbf{F} \) do you need to use the fundamental theorem for line integrals to evaluate the integral.

In order to use the fundamental theorem for line integrals, we need the vector field to be path-independent (i.e. Conservative). That is \( \text{curl} \mathbf{F} = 0 \)

(b) What conditions on the curve \( C \) and/or the vector field \( \mathbf{F} \) do you need to use Green’s Theorem to evaluate the integral.

There curve \( C \) must be a closed curve and be composed of a finite number of smooth paths.

4. Compute curl of the following vector fields. Use the curl to decide whether each vector field is conservative. If the vector field is conservative, find the potential function.

(a) \( \mathbf{F}(x, y, z) = (\sin(x^3) + xz, x - yz, \cos(z^4)) \)

We know that there is a gradient function if and only if the curl is zero. We compute the curl as the cross product

\[
\text{curl } \mathbf{F} = \begin{pmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} (\sin(x^3) + xz) & \frac{\partial}{\partial y} (x - yz) & \frac{\partial}{\partial z} (\cos(z^4))
\end{pmatrix}
= (0 - y, 0, 0 - 0, 1 - x) = (-y, 0, 1 - x) \neq (0, 0, 0)
\]

Which is not zero, so the function does not have a scalar potential function.

(b) \( \mathbf{F}(x, y) = (ye^x, e^x) \)

Again, we test the curl to see if this function has a scalar potential function. We must use the scalar curl in order to perform this test.

\[
\frac{\partial}{\partial x} e^x - \frac{\partial}{\partial y} ye^x = e^x - e^x = 0
\]

So we know this function has a potential function. In order to find the potential function we compute the integrals
\[ \int ye^x \, dx = ye^x \]
\[ \int e^x \, dy = ye^x \]

We look at the terms involving both \( x \) and \( y \), then the terms involving only \( x \) and only \( y \) we find that

\[ f(x, y) = ye^x \]

And we can easily test that this is a scalar potential function.

\[ \nabla f = (ye^x, e^x) \]

5. Use green’s theorem to replace the line integral \( \oint_C (y - \sin(y) \cos(y)) \, dx + 2x \sin^2(y) \, dy \) with a double integral, where \( C \) is the counterclockwise path around the region bounded by \( x = -1, x = 2, y = 4 - x^2 \), and \( y = x - 2 \).

The vector field written as \( adx + bdy \) so we setup our \( F_1 \) and \( F_2 \) functions

\[ F_1(x, y) = y - \sin(y) \cos(y) \]
\[ F_2(x, y) = 2x \sin^2(y) \, dy \]

Then we setup the integral according to Green’s theorem as

\[ \frac{\partial F_1}{\partial y} = 1 \cos^2(y) + \sin^2(x) = 2 \sin^2(y) \]
\[ \frac{\partial F_2}{\partial x} = 2 \sin^2(y) \]

So we setup the double integral

\[ \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA = \iint_R 0 \]
\[ = \int_{-1}^{2} \left( \int_{x-2}^{4-x^2} 0 \right) = 0 \]

So we were able to evaluate the integral, just by writing it out.
6. Find the area between the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \) and the circle \( x^2 + y^2 = 25 \)

A picture is a helpful starting point for this problem.

A naive way to start the problem would be to write

\[
\iint_D dA
\]

But this will be a rather difficult integral to evaluate. We can simplify it using Green’s theorem.

\[
\frac{1}{2} \oint_{\partial D} -y \, dx + x \, dy
\]

In order to proceed we must find parametric equations for the two curves bounding the area. Remember that in order to use Green’s theorem we have to have clockwise orientation.

My choice of parameterizations is:

\[
x_1(t) = (5 \cos t, 5 \sin t) \\
x_2(t) = (3 \cos t, -2 \sin t)
\]

Note that in \( x_2 \) we have had to multiply \( 2 \sin t \) by -1, in order to preserve consistent orientation, i.e. that the region is always on the left as we walk along the boundary. So we can compute the integral.

\[
\frac{1}{2} \oint_{\partial D} -y \, dx + x \, dy = \frac{1}{2} \int_0^{2\pi} ((-5 \sin t)(-5 \sin t) + (5 \cos t)(5 \cos t)) \, dt + \\
+ \frac{1}{2} \int_0^{2\pi} ((2 \sin t)(-3 \sin t) + (3 \cos t)(-2 \cos t)) \, dt
\]

When we reduce this out we have
\[ = \frac{1}{2} \int_{0}^{2\pi} 25dt + \frac{1}{2} \int_{0}^{2\pi} (-6)dt = 19\pi \]

7. Find the area of the region enclosed by the parametric equation

\[ p(\theta) = (\cos(\theta) - \cos^2(\theta), \sin(\theta) - \cos(\theta) \sin(\theta)) \quad \text{for} \ 0 \leq \theta \leq 2\pi \]

The region we are going to find the area of looks like this

We will use Green’s theorem to setup and solve this problem. Recall that the contour integral along a closed path \( C \) of \( F(x, y) = (-y/2, x/2) \) gives the area of the region enclosed by \( C \).

So we will integral \( F \) along \( p(\theta) \).

\[
\int_{p} F(x, y) ds = \int_{p} -y/2 \, dx + x/2 \, dy ds = \int_{0}^{2\pi} \left( (\sin(\theta) - \cos(\theta) \sin(\theta))/2, (\cos(\theta) - \cos^2(\theta))/2 \right) \cdot \left( (2 \cos(t) - 1) \sin(t), \cos(t) - \cos(t)^2 + \sin(t)^2 \right) ds \\
= \int_{0}^{2\pi} 2 \sin(t/2)^4 dt = \frac{3\pi}{2}
\]

So a simple application of Green’s Theorem has given us an answer which would have been very hard to compute without line integrals.

8. Evaluate \( \int_{C} -e^{y} \sin(x) \, dx + e^{y} \cos(x) \, dy + dz \) over the curve \( C \) where \( C \) is the straight line from the point \((0, 0, 0)\) to \((\pi, \pi, 1)\).

It is not difficult to write a parametric equation for \( C \).

\[ P(t) = (\pi t, \pi t, t) \]

Where \( 0 \leq t \leq 1 \). This gives rise to the line integral
\[
\int_C -e^y \sin(x) \, dx + e^y \cos(x) \, dy + dz = \int_C -e^\pi t \sin(\pi t) \, dx + e^\pi t \cos(\pi t) \cdot (\pi, \pi, 1) \, dy + dz \\
= \int_0^1 -\pi e^\pi t \sin(\pi t) + \pi e^\pi t \cos(\pi t) + 1 \, dt
\]

Which gives the final answer of \(-e^\pi\)

A better way to approach this problem would be to notice that the vector field is conservative. We will “notice” this by simply finding a potential function.

\[f(x, y, z) = e^y \cos(x) + z\]

We can check that is really is a potential function since

\[\nabla f = (-e^y \sin(x), e^y \cos(x), 1)\]

Then we can use the fundamental theorem of line integrals to evaluate the integral.

\[
\int_C -e^y \sin(x) \, dx + e^y \cos(x) \, dy + dz = f(\pi, \pi, 1) - f(0, 0, 0) = 1 - e^\pi - 1 = -e^\pi
\]