1. Find the arclength of the path given by the parametric equation \( p(t) \).

\[ p(t) = (e^{-t} \cos(t), e^{-t} \sin(t)) \text{ for } 0 \leq t \leq 2\pi \]

We recall that the formula for computing the arc length between \( p(a) \) and \( p(b) \) is:

\[ \int_a^b \| p'(t) \| \, dt \]

We will begin by computing \( p'(t) \) as

\[ p'(t) = (-e^{-t} \cos(t) - e^{-t} \sin(t), e^{-t} \cos(t) - e^{-t} \sin(t)) \]

Which we can simplify down to

\[ p'(t) = (-e^{-t}(\cos(t) + \sin(t)), e^{-t}(\cos(t) - \sin(t))) \]

Now we can setup and compute the integral

\[
\int_0^{2\pi} \sqrt{(-e^{-t}(\cos(t) + \sin(t)), e^{-t}(\cos(t) - \sin(t))) \cdot (-e^{-t}(\cos(t) + \sin(t)), e^{-t}(\cos(t) - \sin(t)))} \, dt = \\
= \int_0^{2\pi} \sqrt{2e^{-2t}} \, dt = \\
= \sqrt{2} \int_0^{2\pi} e^{-t} \, dt = \\
= \sqrt{2} (1 - e^{-2\pi})
\]

2. Reverse the order of integration in the double integral (you needn’t evaluate the resulting integral).

\[
\int_0^8 \int_{y^2/32}^{y^{1/3}} xy \, dx \, dy
\]

We will begin by drawing a picture of the region.
We are currently integrating $\text{dxdy}$, so after reversing the order of integration we will integrate $\text{dydx}$. Looking at the picture we can easily tell that $x$ can vary between 0 and 2. For a fixed $x$ value we can have $y$ between $x^3$ and $4\sqrt{2}x$. This gives us the integral

$$\int_0^2 \int_{x^3}^{4\sqrt{2}x} xdydx$$

3. Let $W$ be the region in $\mathbb{R}^3$ which is inside the cylinder $y^2 + z^2 = 1$ and bounded by the $yz$-plane and the plane $z + x = 1$.

(a) Set the integral $\int \int \int_W f(x, y, z)\text{dV}$ as a iterated integral taken $\text{dxdzdy}$

the intersection of the $yz$-plane and this volume is a disk of radius 1. Then in $x$ this integral starts at $x = 0$ and continues to $1 - z$. Putting this all together we obtain the bounds on the integral

$$\int_{-1}^1 \int_{\sqrt{1 - y^2}}^{\sqrt{1 - y^2}} f(x, y, z)dxdzdy$$

(b) Setup the integral as an iterated integral with order $\text{dydzdx}$

Integrating in this order is similar. We first note that $x$ varies between 0 and 2. Then allow $z$ to vary from 0 to $1 - x$. Finally we see that $y$ can vary from $-\sqrt{1 - z^2}$ to $\sqrt{1 - z^2}$. This gives the bounds on the integral

$$\int_0^1 \int_0^{\sqrt{1 - z^2}} \int_{-\sqrt{1 - z^2}}^{\sqrt{1 - z^2}} f(x, y, z)dydzdx$$
4. Compute $\int_C \mathbf{F} \cdot ds$ where $\mathbf{F}(x, y, z) = (9x, 4y, z\sqrt{9x^2 + 4y^2})$ and $C$ is parameterized by $C(t) = (2\cos(t), 3\sin(t), 4t^2)$ for $-1 \leq t \leq 2$.

This line integral has no tricks to use in its evaluation, we must proceed with our basic definition of line integral.

To begin we will compute $C'(t)$

$$C'(t) = (-2\sin(t), 3\cos(t), 8t)$$

$$\int_C \mathbf{F} \cdot ds = \int_{-1}^{2} F \circ C(t) \cdot C'(t)dt =$$

$$= \int \left(18\cos(t), 12\sin(t), 4t^2 \sqrt{36\cos(t)^2 + 36\sin(t)^2}\right) \cdot (-2\sin(t), 3\cos(t), 8t) =$$

$$= -36\cos(t)\sin(t) + 36\cos(t)\sin(t) + 192t^3 = 720$$

5. Evaluate the integral (hint: use Green’s Theorem).

$$\oint_C (x^2 + y^2)dx + 2xydy$$

Where $C$ is the curve which follows the parabola $y = x^2$ $(0, 0)$ to $(2, 4)$ then the line from $(2, 4)$ to $(2, 0)$ and finally the line $(2, 0)$ to $(0, 0)$.

We will first compute directly.

The first step is to setup parametric equations. We have three segments, two of them are line segments which are easy to parametrize, the third is a function with 1 dependent variable. So we can easily setup the bounds as

$$x_1(t) = (2t, 4) \quad (0 \leq t \leq 1)$$
$$x_2(t) = (t, t^2) \quad (2 \geq t \geq 0)$$
$$x_3(t) = (0, 2t) \quad (0 \leq t \leq 1)$$
Now we use these in the vector line integral. We setup each line integral the same way

\[ \int_0^1 F(x_1(t)) \cdot x'_1(t) dt = \int_0^1 ((2t)^2 + (4)^2)dx + 2(2t)(4) \cdot (2, 0) = \]
\[ = \int_0^1 8t^2 + 32 dt = \frac{104}{3} \]

\[ \int_0^1 F(x_2(t)) \cdot x'_2(t) dt = \int_0^2 ((t)^2 + (t^2)^2)dx + 2(t)(t^2) \cdot (1, 2t) = \]
\[ = \int_0^2 t^3 + 3t^5 dt = \frac{-104}{3} \]

\[ \int_0^1 F(x_3(t)) \cdot x'_3(t) dt = \int_0^1 (4t^2, 0) \cdot (0, 2) = 0 \]

So we sum these results to get a final result of 0.

Now if were were to use Green’s theorem to do this we would replace this with a double integral. The process goes like this:

First we pull out the \(F_1\) and \(F_2\) from the vector field (or \(M\) and \(N\) depending on which notation you prefer). We compute

\[ \frac{\partial F_1}{\partial y} = 2y \]
\[ \frac{\partial F_2}{\partial x} = 2y \]

Then

\[ \int\int 2y - 2t = \int\int 0 = 0 \]

So we get the same answer.
6. Godzilla has come to terrorize Tokyo. The giant lizard rises from Tokyo bay at time $t = 0$. His path of destruction is given as a function of time by $x(t) = \sin(t) - t \cos(t)$ and $y(t) = \cos(t) + t \sin(t)$ until he returns to the deep at time $t = 2\pi$. If the cost to rebuild a section of Tokyo (in yen) is given as a function of the $x$ $y$ coordinates of the city $C(x, y) = 100000x^2 + 150000y^2$.

(a) How far does Godzilla walk if units are given in kilometers.

The integral we want in order to compute arc-length is:

$$\int_0^{2\pi} \| (x'(t), y'(t)) \| \, dt = \int_0^{2\pi} \sqrt{(t \sin(t))^2 + (t \cos(t))^2} \, dt = \int_0^{2\pi} \sqrt{t^2(1)} \, dt = 2\pi^2$$

(b) How much will it cost to rebuild Tokyo (in yen).

This will be a scalar line integral since the cost function gives a scalar value. Then we use our form for evaluating scalar line integrals to setup the integral as

$$\int_C 100000x^2 + 150000y^2 = \int_0^{2\pi} 100000x(t)^2 + 150000y(t)^2 \| (x'(t), y'(t)) \| \, dt$$

Now all we need to do is expand this all out and integrate.

$$\int_0^{2\pi} 100000x(t)^2 + 150000y(t)^2 \| (x'(t), y'(t)) \| \, dt$$

$$= \int_0^{2\pi} 100000 \left(\sin(t) - t \cos(t)\right)^2 + 150000 \left(\cos(t) + t \sin(t)\right)^2 \cdot t$$

$$= -25000t(-5t^2 + 1) + (t^2 - 1) \cos(2t) - 2t \sin(2t)$$

$$= 25000\pi^2(3 + 20\pi^2) = 4.944 \times 10^7$$