The purpose of this worksheet is to review the method for answering certain types of questions.

1. **Directions:** Write an equation for the function described, then graph the function.

   **Method** for graphing: Do not graph by plotting points! Rather, begin by plotting simpler functions, then perform transformations to obtain the given function. (You may check by plotting a few points at the end to make sure your graph makes sense.)

   (a) The function is \( f(x) = x^3 \),

   - reflected through the \( x \)-axis
   - shrunk by a factor of \( 1/2 \)
   - translated 2 units left and 4 units down

   **Equation:**
   
   Start with \( f(x) = x^3 \).
   Reflect through \( x \)-axis: \( f(x) = -x^3 \)
   Shrink by \( 1/2 \): \( f(x) = -\frac{1}{2}x^3 \)
   Translate 2 left and 4 down:
   
   \[
   f(x) = -\frac{1}{2}(x + 2)^3 - 4
   \]

   The graph is a cubic decreasing everywhere and flattening out near the point \((-2, -4)\). The intercepts are \((0, -8)\) and \((-4, 0)\).

   (b) The function is \( f(x) = \sqrt{x} \),

   - reflected through the \( y \)-axis
   - stretched by a factor of 4
   - translated 3 units right and 1 unit up

   **Equation:**
   
   Start with \( f(x) = \sqrt{x} \).
   Reflect through \( y \)-axis: \( f(x) = \sqrt{-x} \)
   Stretch by 4: \( f(x) = 4\sqrt{-x} \)
   Translate 3 right and 1 up:
   
   \[
   f(x) = 4\sqrt{-(x - 3)} + 1 = 4\sqrt{3 - x} + 1
   \]

   The graph is a square root function starting at \((3, 1)\) and opening in the negative \( x \) direction. The \( y \)-intercept is \((0, 4\sqrt{3} + 1)\).
2. **Directions**: Find \( f \circ g(x) \), and find its domain.

**Method** for finding the domain:
(i) Find the points \( x \) in the domain of \( g(x) \).
(ii) Find the points \( g(x) \) in the domain of \( f(x) \).
(iii) Combine the restrictions from (i) and (ii) to find the domain of \( f \circ g(x) \).

(a) \( f(x) = \frac{1}{x+1}, g(x) = 3x + 5 \)

\[
\begin{align*}
f \circ g(x) &= f(g(x)) = f(3x + 5) \\
&= \frac{1}{(3x + 5) + 1} = \frac{1}{3x + 6}
\end{align*}
\]

Domain of \( g \) is \( \{ x : x \text{ is any real number} \} \)

Domain of \( f \) is \( \{ g(x) : g(x) \neq -1 \} \)

Solve for the restriction this gives on \( x \):

\[
g(x) \neq -1 \\
3x + 5 \neq -1 \Rightarrow x \neq -2
\]

Domain of \( f \circ g \) is \( \{ x : x \neq -2 \} \)

(b) \( f(x) = \frac{1}{2x - 1}, g(x) = \sqrt{x} \)

\[
\begin{align*}
f \circ g(x) &= f(g(x)) = f(\sqrt{x}) \\
&= \frac{1}{2\sqrt{x} - 1}
\end{align*}
\]

Domain of \( g \) is \( \{ x : x \geq 0 \} \)

Domain of \( f \) is \( \{ g(x) : g(x) \neq \frac{1}{2} \} \)

Solve for the restriction this gives on \( x \):

\[
g(x) \neq \frac{1}{2} \\
\sqrt{x} \neq \frac{1}{2} \\
x \neq \frac{1}{4}
\]

Domain of \( f \circ g \) is \( \{ x : x \geq 0, x \neq \frac{1}{4} \} \)
3. **Directions:** Find the inverse of each function, if an inverse exists.

**Method:**
(i) Check whether the function is one-to-one by graphing it and using the Horizontal Line Test.
(ii) Go through the four-step method for finding an inverse. (Replace \( f(x) \) with \( y \), switch \( x \) and \( y \), solve for \( y \), replace \( y \) with \( f^{-1}(x) \).)
(iii) Check that the domain of \( f^{-1} \) is the range of \( f \) and the domain of \( f \) is the range of \( f^{-1} \). If not, make the necessary restrictions on the domain of \( f^{-1} \).

(a) \( f(x) = \sqrt{x - 3} \)

(i) \( f \) is one-to-one since its graph passes the Horizontal Line Test.
(ii)

\[
\begin{align*}
f(x) & = \sqrt{x - 3} \\
y & = \sqrt{x - 3} \\
x & = \sqrt{y - 3} \\
x^2 & = y - 3 \\
x^2 + 3 & = y \\
f^{-1}(x) & = x^2 + 3
\end{align*}
\]

(iii) The domain of \( f \) is \( \{ x : x \geq 3 \} \), which agrees with the range of \( f^{-1} \). The domain of \( f^{-1} \) is all real numbers, but the range of \( f \) is positive real numbers. That means that we have to restrict the domain of \( f^{-1} \) to be only positive real numbers.

Solution: \( f^{-1}(x) = x^2 + 3 \) with domain \( \{ x : x \geq 0 \} \).

(b) \( f(x) = -(x + 1)^2 \)

(i) The graph of \( f \) is a parabola, so it does not pass the Horizontal Line Test. That means that \( f \) is not one-to-one, so it does not have an inverse function.
4. **Directions:** Determine whether $f$ and $g$ are inverse functions.

**Method:**
(i) Check that the domain of $f$ is the range of $g$, and that the domain of $g$ is the range of $f$.
(ii) Check that $f \circ g(x) = x$ and $g \circ f(x) = x$.

(a) $f(x) = x^2 + 1, g(x) = \sqrt{x - 1}$

(i) The domain of $f$ is all real numbers, and the range of $g$ is the positive real numbers. Since these do not agree, $f$ and $g$ are not inverse functions.

(b) $f(x) = 2x - \frac{1}{3}, g(x) = \frac{3x + 1}{6}$

(i) The domain and range of $f$ and of $g$ are all real numbers.
(ii)

\[
\begin{align*}
  f \circ g(x) &= f(g(x)) = f\left(\frac{3x + 1}{6}\right) \\
  &= 2\left(\frac{3x + 1}{6}\right) - \frac{1}{3} \\
  &= \frac{6x + 2}{6} - \frac{1}{3} \\
  &= x + \frac{2}{6} - \frac{1}{3} = x \\

  g \circ f(x) &= g(f(x)) = g\left(2x - \frac{1}{3}\right) \\
  &= \frac{3\left(2x - \frac{1}{3}\right) + 1}{6} \\
  &= \frac{6x - 1 + 1}{6} \\
  &= \frac{6x}{6} = x
\end{align*}
\]

This shows that $f$ and $g$ are inverse functions.