Math 2263
Midterm 1 Solutions

1. The height of a hill is given by the function $f(x, y) = x^2 - y^2 - 2x + 2y + 2$.

(a) Locate all critical points for this function, and classify each critical point as a local maximum, local minimum, or saddle point.

$$f_x = 2x - 2 = 0 \Rightarrow x = 1$$
$$f_y = -2y + 2 = 0 \Rightarrow y = 1$$

The only critical point is the point $(1, 1)$. I will use the Second Derivative Test to determine whether this is a local maximum, local minimum, or saddle point.

$$f_{xx} = 2$$
$$f_{xy} = 0$$
$$f_{yy} = -2$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2(-2) - 0 = -4 < 0$$

Since $D < 0$, the point $(1, 1)$ must be a saddle point.

(b) Consider the graph $z = f(x, y)$. What is the name of this quadric surface? (Hint: Complete the square to rewrite the equation for the surface.)

We can rewrite the equation for the graph as $z - 2 = (x - 1)^2 - (y - 1)^2$, so this is a hyperbolic paraboloid.

(c) A raindrop falls at the point $(1, 0)$ and rolls downhill in the direction of steepest descent. Give a unit vector $u$ describing this direction.

The direction of steepest descent is the direction of $-\nabla f(1, 0) = \langle 0, -2 \rangle$. The unit vector in this direction is $\langle 0, -1 \rangle$. 

2. Determine whether the function $f$ is continuous at $(0, 0)$.

$$f(x, y) = \begin{cases} \frac{2x^2y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

In order for the function to be continuous at $(0, 0)$, I need

$$\lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^4+y^2} = f(0, 0).$$

To determine if the limit on the left exists, I will try approaching $(0, 0)$ along different curves and see if the solutions agree.

Along the curve $y = 0$, we have

$$\lim_{x \to 0} \frac{0}{x^4} = 0.$$

Along the curve $y = x^2$, we have

$$\lim_{x \to 0} \frac{2x^2 \cdot x^2}{x^4 + x^4} = \frac{2x^4}{2x^4} = 1.$$

Since I obtained different values along different curves,

$$\lim_{(x,y) \to (0,0)} \frac{2x^2y}{x^4+y^2}$$

does not exist. It follows that $f$ is not continuous at $(0, 0)$. 
3. Consider the planes $x + 2y - z - 4 = 0$ and $3x + 2y + 6 = 0$.
   (a) Find the angle between these planes.

   $\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2| \cos \theta$
   $<1, 2, -1> \cdot <3, 2, 0> = \sqrt{6}\sqrt{13} \cos \theta$
   $\theta = \cos^{-1}(\frac{7}{\sqrt{78}})$

(b) Find symmetric equations for the line where these planes intersect.

   $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = <2, -3, -4>$
   $\mathbf{r}_0 = <0, -3, 10>$

   $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
   $\frac{x}{2} = \frac{y + 3}{-3} = \frac{z - 10}{-4}$
4. Suppose we have a function \( p = f(q, r, s) \), where \( q(x, y) = \frac{y}{x} \), \( r(x, y) = e^{xy} \), and \( s(x, y) = x - y \). Given that

\[
\frac{\partial f}{\partial q}(4, e, -3/2) = 2e, \quad \frac{\partial f}{\partial r}(4, e, -3/2) = 4, \quad \text{and} \quad \frac{\partial f}{\partial s}(4, e, -3/2) = e,
\]

find \( \frac{\partial p}{\partial x} \) when \( (x, y) = (1/2, 2) \).

\[
p_x = p_q q_x + p_r r_x + p_s s_x
\]

\[
= (2e)(-8) + (4)(2e) + (e)(1) = -7e
\]
5. Find the absolute maximum and minimum of the function

\[ f(x, y) = 3x + 5y + z \]

subject to the constraint \( x^2 + y^2 + z^2 = \frac{5}{7} \).

We have \( g(x, y, z) = x^2 + y^2 + z^2 \).

\[ \nabla f = \lambda \nabla g \]

\[ < 3, 5, 1 > = \lambda < 2x, 2y, 2z > \]

\[ \frac{3}{2\lambda} = x, \quad \frac{5}{2\lambda} = y, \quad \frac{1}{2\lambda} = z \]

Replacing these values into the equation \( x^2 + y^2 + z^2 = \frac{5}{7} \), we can solve for \( \lambda \).

\[ \frac{9}{4\lambda^2} + \frac{25}{4\lambda^2} + \frac{1}{4\lambda^2} = \frac{5}{7} \]

\[ \frac{49}{4} = \lambda^2 \Rightarrow \lambda = \pm \frac{7}{2} \]

So the possible maximum and minimum points are \((3/7, 5/7, 1/7)\) when \( \lambda = 7/2 \), and \((-3/7, -5/7, -1/7)\) when \( \lambda = -7/2 \). The absolute maximum value is \( f(3/7, 5/7, 1/7) = 5 \) and the absolute minimum value is \( f(-3/7, -5/7, -1/2) = -5 \).
6. Let $g(x, y) = xe^{xy - 2}$.

(a) Find the tangent plane for $g$ at the point $(1, 2)$. Write the plane in the form $z = Ax + By + C$.

$g(1, 2) = 1$
$g_x(1, 2) = xy e^{xy - 2} + e^{xy - 2} |_{(1,2)} = 3$
$g_y(1, 2) = x^2 e^{xy - 2} |_{(1,2)} = 1$

\[
z = z_0 + f_x(x - x_0) + f_y(y - y_0) \]
\[= 1 + 3(x - 1) + (y - 2) = 3x + y - 4\]

(b) Using a linear approximation, estimate the value of $f$ at the point $(1.2, .9)$. (A calculator tells us that this value is approximately .478223.)

\[
f(x, y) \approx 3x + y - 4\]
\[f(1.2, .9) \approx 3(1.2) + .9 - 4 = .5\]