Midterm 2 Solutions

1. Consider the integral

\[ \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \sqrt{x^2 + y^2} e^{x^2+y^2} \, dz \, dy \, dx \]

(a) Rewrite this integral in cylindrical coordinates, \( r, \theta, \) and \( z. \)

\[ \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{4-r^2}} r^2 e^{r^2} \, dz \, dr \, d\theta \]

(b) Rewrite this integral in spherical coordinates, \( \rho, \theta, \) and \( \phi. \)

\[ \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_0^{\rho^2} \rho^3 \sin^2 \phi \, e^{\rho^2 \sin^2 \phi} \, d\rho \, d\phi \, d\theta \]
2. Evaluate the integral.

\[ \int_0^6 \int_{x/2}^3 \frac{1}{y^2 + 1} \, dy \, dx \]

Rather than trying to find an antiderivative for the integrand with respect to \( y \), we will change the order of integration.

\[ \int_0^3 \int_0^{2y} \frac{1}{y^2 + 1} \, dx \, dy = \int_0^3 \frac{1}{y^2 + 1} \left. \right|_{x=0}^{x=2y} \, dx \]

\[ = \int_0^3 \frac{2y}{y^2 + 1} \, dy \]

Substitute using \( u = y^2 + 1 \).

\[ \ln(y^2 + 1) \bigg|_0^3 = \ln 10 - \ln 1 = \ln 10 \]
3. Let $X$ and $Y$ be random variables with joint density function $f$.

\[
f(x, y) = \begin{cases} 
  Cxy & \text{if } 0 \leq x \leq 1, \ 1 \leq y \leq 3 \\
  0 & \text{otherwise}
\end{cases}
\]

(a) Find the value of the constant $C$.

\[
1 = \int_1^3 \int_0^1 Cxy \, dx \, dy = C \int_1^3 (1/2)y \bigg|_{y=1}^{y=3} dy
\]

\[
= C \int_1^3 \frac{1}{2} y \, dy
\]

\[
= \frac{1}{4} y^2 \bigg|_{y=1}^{y=3}
\]

\[
C \left( \frac{9}{4} - \frac{1}{4} \right) = 2C
\]

So $C = 1/2$.

(b) Find the probability that $Y$ is no more than one larger than $X$.

\[
P(Y \leq X + 1) = \int_0^1 \int_1^{x+1} (1/2)xy \, dy \, dx
\]

\[
= \int_0^1 \frac{1}{4} xy^2 \bigg|_{y=1}^{y=x+1} dx
\]

\[
= \int_0^1 \frac{1}{4} x^3 + \frac{1}{2} x^2 dx
\]

\[
= \frac{1}{16} x^4 + \frac{1}{6} x^3 \bigg|_{x=0}^{x=1} = \frac{11}{48}
\]
4. Use a double integral to find the area inside the curve \( r = 1 + \sin \theta \) and outside the curve \( r = 1 \). (It maybe be helpful to know that \( \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos(2\theta) \).

\[
\int_{0}^{\pi} \int_{1}^{1 + \sin \theta} r \, dr \, d\theta = \int_{0}^{\pi} \left( \frac{1}{2} \right) (1 + \sin \theta)^2 \, d\theta
\]

\[
= \int_{0}^{\pi} \left( \frac{1}{2} \right) (1 + \sin \theta + 1/2 - 1/2 \cos(2\theta)) \, d\theta
\]

\[
= \left[ \frac{3}{4} \theta - \cos \theta - \frac{1}{2} \sin(2\theta) \right]_{0}^{\pi}
\]

\[
= \left( \frac{3}{4} \pi - (-1) - 0 \right) - (0 - 1 - 0)
\]

\[
= 2 + \frac{3}{4} \pi
\]
5. Use spherical coordinates to evaluate the integral

\[
\iiint_E e^{(x^2+y^2+z^2)^{3/2}} \, dV,
\]

where \( E \) is the solid bounded by the hemispheres \( z = \sqrt{4 - x^2 - y^2} \) and \( z = \sqrt{1 - x^2 - y^2} \), and inside the cone \( z = \sqrt{x^2 + y^2} \).

\[
\int_0^{\pi/4} \int_0^{2\pi} \int_1^2 e^{\rho^3} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi/4} \int_0^{2\pi} (1/3) e^{\rho^3} \sin \phi \big|_{\rho=1}^{\rho=2} \, d\theta \, d\phi
\]

\[
= \frac{e^8 - e}{3} \int_0^{\pi/4} \int_0^{2\pi} \sin \phi \, d\theta \, d\phi
\]

\[
= \frac{2\pi}{3} (e^8 - e) \int_0^{\pi/4} \sin \phi \, d\phi
\]

\[
= \frac{2\pi}{3} (e^8 - e) (- \cos \phi) \big|_0^{\pi/4}
\]

\[
= \frac{2\pi}{3} (e^8 - e) (1 - \frac{1}{\sqrt{2}})
\]
6. Find the mass $E$, where $E$ is the region in $\mathbb{R}^3$ inside $x^2 + y^2 = 1$, above $z = 0$ and below the cone $z^2 = 4x^2 + 4y^2$. The mass per unit volume at a point $(x, y, z)$ in $E$ is $\rho(x, y, z) = z(x^2 + y^2)$.

\[
\int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 z \, r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (1/2)z^2 r^3 \bigg|_{z=0}^{2r} \, dr \, d\theta = \int_0^{2\pi} \int_0^1 2r^5 \, dr \, d\theta = \int_0^{2\pi} (1/3)r^6 \bigg|_0^1 \, d\theta = \int_0^{2\pi} (1/3) \, d\theta = 2\pi/3
\]