Midterm 3 Solutions

1. Find the integral of the vector field
\[ \mathbf{F}(x, y, z) = (2x + y^{14}z^{19})\mathbf{i} + 14xy^{13}z^{19}\mathbf{j} + 19xy^{14}z^{18}\mathbf{k} \]
along the line segment from (1, 0, 2) to (2, 1, −1).
We will use the Fundamental Theorem of Line Integrals since \( \mathbf{F} \) is the
gradient field of the function \( f(x, y, z) = xy^{14}z^{19} + x^2 \).
\[
\int_C \nabla f(x, y, z) \, ds = f(2, 1, -1) - f(1, 0, 2) = (-2 + 4) - (1) = 1
\]

2. Evaluate the integral
\[
\int_C 6xe^y \, dx,
\]
where \( C \) is the curve \( x = e^y + 1 \) from \((1 + \frac{x}{e}, -1) \) to \((2, 0)\).
We can parametrize the curve by
\[
\mathbf{r}(t) = <e^t + 1, t>
\]
with \(-1 \leq t \leq 0\). Then \( x(t) = e^t + 1 \), so \( x'(t) = e^t \).
\[
\int_C 6xe^y \, dx = \int_{-1}^{0} 6(e^t + 1)e^t e^t \, dt
\]
\[
= \int_{-1}^{0} (6e^{3t} + 6e^{2t}) \, dt
\]
\[
= 2e^{3t} + 3e^{2t}\bigg|_{-1}^{0}
\]
\[
= 5 - 2e^{-3t} - 3e^{-2t}
\]

3. Consider the vector field \( \mathbf{F} = zi + j + (\ln y)k \).
(a) Find \( \text{curl}(\mathbf{F}) \) and \( \text{div}(\mathbf{F}) \).
\[
\nabla \times \mathbf{F} = < \frac{1}{y} - 0, 1 - 0, 0 - 0 > = < \frac{1}{y}, 1, 0 >
\]
\[
\nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0
\]

(b) Is \( \mathbf{F} \) conservative? If so, find a function \( f(x, y) \) such that \( \nabla f = \mathbf{F} \).
Since \( \text{curl}(\mathbf{F}) \neq 0 \), \( \mathbf{F} \) is not conservative.
4. Compute the integral
\[ \int_C xy^2 \, ds, \]
where \( C \) is the curve \( x^2 + y^2 = 9 \) from the point \((0, -3)\) to \((0, 3)\) through the right half of the plane.
We can parametrize the curve by
\[ \mathbf{r}(t) = < 3 \cos t, 3 \sin t >, \quad -\pi/2 \leq t \leq \pi/2. \]
\[
\int_C xy^2 \, ds = \int_{-\pi/2}^{\pi/2} (3 \cos t)(9 \sin^2 t) \sqrt{9 \cos^2 t + 9 \sin^2 t} \, dt \\
= \int_{-\pi/2}^{\pi/2} 81 \cos t \sin^2 t \, dt \\
= -27 \sin^3 t \bigg|_{-\pi/2}^{\pi/2} = 54
\]

5. Compute the integral
\[ \int_C \left( (-4x^2y + e^{x^3}) \mathbf{i} + (4xy^2 + \sqrt{y^5 + 2}) \mathbf{j} \right) \cdot d\mathbf{r} \]
where \( C \) is the path consisting of the line segment from \((0, 0)\) to \((2, 0)\), the curve \( x^2 + y^2 = 4 \) from \((2, 0)\) to \((0, 2)\), and the line segment from \((0, 2)\) to \((0, 0)\).
\[
\int_C \left( (-4x^2y + e^{x^3}) \mathbf{i} + (4xy^2 + \sqrt{y^5 + 2}) \mathbf{j} \right) \cdot d\mathbf{r} \\
= \int_D \frac{\partial}{\partial x} (4xy^2 + \sqrt{y^5 + 2}) - \frac{\partial}{\partial y} (-4x^2y + e^{x^3}) \, dA \\
= \int_D (4x^2 + 4y^2) \, dA \\
= \int_0^2 \int_0^{\pi/2} 4r^2 \, r \, dr \, d\theta \\
= \int_0^{\pi/2} r^4 |_0^2 \, d\theta \\
= \int_0^{\pi/2} 16 \, d\theta = 8\pi
\]
6. Find the work done by the force field \( \mathbf{F}(x, y, z) = y\mathbf{i} + z^2\mathbf{j} + x\mathbf{j} \) when moving an object in a straight line path from \((1, -1, 0)\) to \((3, 2, 5)\).

The line can be parametrized as follows.

\[
\mathbf{r}(t) = (1-t) <1, -1, 0> + t <3, 2, 5> = <1+2t, 3t-1, 5t>, \quad 0 \leq t \leq 1
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 <3t - 1, 25t^2, 1 + 2t> \cdot <2, 3, 5> \ dt
\]

\[
= \int_0^1 (75t^2 + 16t + 3) \ dt
\]

\[
= 25t^3 + 8t^2 + 3t \bigg|_0^1 = 36
\]