Quiz 9: 16.7 Solutions

Math 2263
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1. Find $\iint_S (xz + 3x^2 + 2xy) \, dS$, where $S$ is the region in the plane $3x + 2y + z = 6$ which lies in the first octant.

Start with a parametrization of the plane.

$$\mathbf{r}(x, y) = <x, y, 6 - 3x - 2y>$$

To find bounds on the parameters $x$ and $y$, I look in the $xy$-plane. The vertices of this triangular piece of the plane are at $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$, so the corresponding triangle in the $xy$-plane has vertices $(2, 0), (0, 3),$ and $(0, 0)$. This gives bounds on the parameters.

$$0 \leq y \leq 3 - 3x/2$$ and $$0 \leq x \leq 2$$

Next I compute $|\mathbf{r}_x \times \mathbf{r}_y|$.

$$\mathbf{r}_x = <1, 0, -3>$$

$$\mathbf{r}_y = <0, 1, -2>$$

$$|\mathbf{r}_x \times \mathbf{r}_y| = |<3, 2, 1>| = \sqrt{14}$$

Now I can set up the integral.

$$\iint_S (xz + 3x^2 + 2xy) \, dS = \int_0^2 \int_0^{3-3x/2} [x(6-3x-2y) + 3x^2 + 2xy] \sqrt{14} \, dy \, dx$$

$$= \sqrt{14} \int_0^2 \int_0^{3-3x/2} (6x) \, dy \, dx$$

$$= \sqrt{14} \int_0^2 (18x - 9x^2) \, dx$$

$$= \sqrt{14}(9x^2 - 3x^3)|_0^2 = 12\sqrt{14}$$

2. Find the flux of the field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z^2\mathbf{k}$ across the surface $S$, where $S$ is the part of the surface $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 4$, oriented downwards.
Start with a parametrization of the surface.

\[ \mathbf{r}(r, \theta) = < r \cos \theta, r \sin \theta, r >, \]

where \( 0 \leq r \leq 2 \), \( 0 \leq \theta \leq 2\pi \).

\( \mathbf{r}_r \times \mathbf{r}_\theta = < -r \cos \theta, -r \sin \theta, r > \)

We can check orientation by making sure that the \( z \)-coordinate is negative. Since it is not, we’ll have to use

\( \mathbf{r}_\theta \times \mathbf{r}_r = < r \cos \theta, r \sin \theta, -r > \)

instead.

\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^2 < r^2 \cos \theta, r^2 \sin \theta, -r^2 > \cdot < r \cos \theta, r \sin \theta, -r > \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 (r^3 \cos^2 \theta + r^3 \sin^2 \theta + r^3) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 2r^3 \, dr \, d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} r^4 \bigg|_{r=0}^{r=2} \, d\theta
\]

\[
= \int_0^{2\pi} 8 \, d\theta = 16\pi
\]