Bob the beetle, Lily the ladybug, and Sal the Spider all live happily in the $xy$ plane. At time $t$ Bob’s position is given by the parametric equation

$$b(t) = \left( \cos(\pi t), \sin(\pi t) \right).$$

Simultaneously, Lily’s position is given by

$$l(t) = \left( t - 1, t \right).$$

1. Do Bob and Lily’s paths ever cross? Do Bob and Lily ever meet?

Solution: Bob is travelling in a circle of radius one centered at the origin. Lily is travelling along the line $y = x + 1$. These paths intersect at the points $(-1, 0)$ and $(0, 1)$. In order for Bob and Lily to meet, they must both reach one of these points at the same time $t$. Lily is at the point $(-1, 0)$ when $t = 0$, but Bob doesn’t arrive there until $t = 1$. Lily arrives at the point $(0, 1)$ when $t = 1$, but Bob passed that point when $t = 1/2$. Even though Bob and Lily’s paths intersect, the bugs never meet.

2. At time $t = 0$ Sal begins running in a straight line from the point $(-1/3, 2/3)$. By the time $t = 1$, she has reached the point $(1/3, 4/3)$. Find a parametric equation $s(t)$ for Sal’s path. (Hint: use the equation for a parametric line between from $\mathbf{a}$ to $\mathbf{b}$,

$$s(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) = t\mathbf{b} + (1 - t)\mathbf{a}.$$

Solution: In the equation $l(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, $l(0) = \mathbf{a}$ and $l(1) = \mathbf{b}$, so we can use $\mathbf{a} = (-1/3, 2/3)$ and $\mathbf{b} = (1/3, 4/3)$. This gives the equation

$$l(t) = (-1/3, 2/3) + t(2/3, 2/3).$$

3. Does Sal cross paths with Bob or Lily? Assuming Sal continues to run in a straight line, will she ever run into Bob or Lily?
Solution: The direction vector for Sal’s path is \((2/3, 2/3)\), so the slope is
\[
m = \frac{\text{rise}}{\text{run}} = \frac{2/3}{2/3} = 1.
\]
Since Sal’s path goes through the point \((-1/3, 2/3)\), the equation for her path must be
\[
y - 1/2 = 1(x + 1/2)
\]
\[
y = x + 1.
\]
This is identical to the path that Lily follows! To check whether Sal and Lily ever meet, we’ll check for a \(t\)-value where the \(x\)-coordinates for Sal’s and Lily’s paths agree, that is, \(t - 1 = 2/3t - 1/3\). This occurs at \(t = 2\), so Sal and Lily meet at the point \((1, 2)\) at time \(t = 2\). As above, Sal and Bob’s paths intersect at \((-1, 0)\) and \((0, 1)\). Sal would have reached the point \((-1, 0)\) when \(t = -1\) if she’d been moving along her path before \(t = 0\), but Bob reaches this point when \(t = 1\). Sal reaches \((0, 1)\) when \(t = 1/2\), which is exactly when Bob gets there!

4. Find the matrix of partial derivatives of the function
\[
F(x, y, z) = (ze^{x^2+y^2} + xy, \cos(x^3y^2z^4))
\]
Solution: We see that \(F : \mathbb{R}^3 \to \mathbb{R}^2\) so we know that we will have a \(2 \times 3\) matrix, and that we will need to compute 6 partial derivatives. To begin we will rewrite
\[
F(x, y, z) = (f_1(x, y, z), f_2(x, y, z))
\]
Where
\[
f_1(x, y, z) = ze^{x^2+y^2} + xy
\]
\[
f_2(x, y, z) = \cos(x^3y^2z^4)
\]
So we will compute all the partial derivatives independently.

\[
\begin{align*}
\frac{\partial f_1}{\partial x} &= y + 2xze^{x^2+y^2} \\
\frac{\partial f_1}{\partial y} &= x + 2yze^{x^2+y^2} \\
\frac{\partial f_1}{\partial z} &= e^{x^2+y^2} \\
\frac{\partial f_2}{\partial x} &= -3x^2y^2z^4 \sin(x^3y^2z^4) \\
\frac{\partial f_2}{\partial y} &= -2x^3yz^4 \sin(x^3y^2z^4) \\
\frac{\partial f_2}{\partial z} &= -4x^3y^2z^3 \sin(x^3y^2z^4)
\end{align*}
\]

We then put these partial derivatives into the matrix of partial derivatives as follows

\[
\begin{pmatrix}
y + 2xze^{x^2+y^2} & x + 2yze^{x^2+y^2} & e^{x^2+y^2} \\
-3x^2y^2z^4 \sin(x^3y^2z^4) & -2x^3yz^4 \sin(x^3y^2z^4) & -4x^3y^2z^3 \sin(x^3y^2z^4)
\end{pmatrix}
\]

5. Mathematical physicists have determined that, in order for a function \(u(x, t)\) to describe the flow of heat through a metal bar along the \(x\)-axis, the function must satisfy the heat equation: \(\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}\).

You are an experimental physicist collecting data about the heat flow, and you find that the function \(u(x, t) = e^{-k^2t} \sin(x)\) models the flow of heat through the bar. Is this consistent with the theory?

Solution: We begin by computing the partial derivatives used in the heat equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -e^{-k^2t} \sin(x) \\
\frac{\partial^2 u}{\partial x^2} &= -e^{-k^2t}k^2 \sin(x)
\end{align*}
\]

6. Let \(f(x, y) = x^2 + \frac{1}{3}y^2 - 2x\) Find a point on the graph \(z = f(x, y)\) where the tangent plane is horizontal.

Solution: We recall from in class, lab, textbooks, and folklore that we can write the normal vector at a point as
\[
\hat{n} = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1)
\]

So for our purposes we will have

\[
\hat{n} = (-2x + 2, -y, 1)
\]

So all we need to find a point \((x, y)\) where \(\hat{n} = (0, 0, 1)\). The choice of \((1, 0)\) is one such point. \(f(1, 0) = -1\) meaning that a point where the tangent plane is horizontal will be \((1, 0, -1)\).

Another good way to approach this part of the problem was to say “If the plane is horizontal then \(\frac{\partial f}{\partial x} = 0\) and \(\frac{\partial f}{\partial y} = 0\). This gives the same solution.

7. Let \(f(x, y) = x/y + y/x\). Using a linear approximation about the point \((1/2, 1/4)\), estimate the value of \(f(.48, .3)\).

Solution: We begin by recalling that the tangent plane gives a linear approximation to \(f(x, y)\) at a point, so we first compute the matrix of partial derivatives.

\[
Df(x, y) = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{y} - \frac{y}{x^2} & \frac{1}{x} - \frac{x}{y^2}
\end{pmatrix}
\]

Evaluating this at the point \((1/2, 1/4)\) we have

\[
Df(1/2, 1/4) = \begin{pmatrix}
1/4 - 1/2 & 1/2 - 1/4
\end{pmatrix}
= (3, -6)
\]

Now recall the form of the tangent plane \(L(x, y)\) at the point \((a, b)\).

\[
L(x, y) = f(a, b) + Df(a, b) \cdot (x - a, y - b)
\]

and for the point \((1/2, 1/4)\) this becomes
\[ L(x, y) = f(a, b) + Df(a, b) \cdot (x - a, y - b) = \\
= f(1/2, 1/4) + Df(1/2, 1/4) \cdot (x - 1/2, y - 1/4) \\
= \frac{5}{2} + (3, -6) \cdot (x - 1/2, y - 1/4) \\
= \frac{5}{2} + 3x - \frac{3}{2} - 6y + \frac{6}{4} \\
= \frac{5}{2} + 3x - 6y \\
\]

Now to estimate the value of \( f(.48, .3) \) we compute

\[ L(.48, .3) = \frac{5}{2} + 3 \cdot (.48) - 6 \cdot (.3) = 2.14 \]

Using a computer we can compute the exact value to be 2.225, and we see that our estimate is fairly close.