Worksheet 4: 2.4-2.6

Names

1. On a far-away planet, the composition of nitrogen, oxygen, and argon in the air is given by

\[ f(x, y, z) = (x^2, xy - z^2, 2x^2y^3), \]

where \( f_1(x, y, z) \) gives the concentration of nitrogen, \( f_2(x, y, z) \) gives the concentration of oxygen, and \( f_3(x, y, z) \) gives the concentration of argon. An alien bird follows the path \( p(t) = (t^2, 2t - 1, 4 - t) \).

(a) Find the derivative matrix \( D(f \circ p) \) using the chain rule.
(b) Check your solution to (a) by first finding \( f \circ p \), then taking the derivative of the composition.
(c) At time \( t = 1 \), does the alien bird experience a greater increase in oxygen or argon? Explain.

2. Stranded on a makeshift raft, Odysseus is swept into the whirlpool Charybdis! Charybdis creates her whirlpool by applying a centripetal force with magnitude

\[ f(x, y) = x^2 + y^2, \]
toward the center of the whirlpool. Odysseus’ raft follows the elliptical path

\[ p(t) = (2 \cos(\pi t), 3 \sin(\pi t)). \]

(a) Sketch a few level curves of \( f(x, y) \). Sketch the path of Odysseus’ raft on the same set of axes. Is the inward force constant along the entire path of Odysseus’ raft?
(b) When \( t = 1 \), in what direction is the raft traveling? If Odysseus manages to escape Charybdis’ maw and continue to travel in this direction, what line will he follow?
(c) Show that the rate of change of \( f(x, y) \) in the direction of the path from part (b) is equal to 0. (You will need to compute a directional derivative.) Why is this? Explain in terms of the level curves of \( f(x, y) \).
(d) At \( t = 7/4 \), Odysseus wants to move as quickly as possible to a place where Charybdis’ force is lower. In what direction should he paddle?
3. An itsy bitsy spider is at the point $x = 2, y = -1, z = 7$ in a parabolic water spout with height $z = x^2 + 3y^2$. In order to get to the top of the spout as fast as possible (before the rain comes down to wash the spider out), in which direction should the spider set out? Should it follow a straight line path from then on?

4. The mighty nations of Notwen and Zinbiel have been at war for many years. Through years of painstaking research, Notwen-ian scientists have determined that the Zinbiel workers require both wood (denoted by $W$) and metal (denoted by $M$) to build a cannon, and that the rate of cannon production (in cannons per month) is given by the equation

$$P = 5M^{1/2}W^{1/2}$$

Historically the Zinbielians have used 16 units of wood and 4 units metal.

(a) Historically, how many cannons are produced per month?

(b) One month, Notwen spies report that the Zinbiels are producing 2 additional units of wood per month and 10 additional cannons per month. Deduce how the production of metal has changed.

(c) Years later, wood and metal production levels have return to $W = 16, M = 4$ when a treaty is signed calling for Zinbiel to produce half the number of cannons. If spies report that both metal and wood production have been decreased by 2 units per month, can Notwen be sure Zinbiel is honoring the treaty? (Could they be producing more cannons than the treaty allows?)