Due date: Friday, July 26

**Short Answer**
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. Find \( f' \) when \( f : [0, 1] \to \mathbb{R} \) is defined as follows.
   
   (a) \( f(x) = \int_0^x \sqrt{t^2 + 1} \, dt \)
   
   (b) \( f(x) = \int_x^1 \cos \frac{1}{t+1} \, dt \)

2. For each integral, identify the functions \( f \) and \( \phi \) to apply the Change of Variables Theorem (5.18), then evaluate each integral. You may use rules of basic integration without proof.
   
   (a) \( \int_0^3 x \sqrt{1 + x^2} \, dx \)
   
   (b) \( \int_1^4 \frac{(\sqrt{x} + 2)^3}{\sqrt{x}} \, dx \)

**Proofs**
For the remaining problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Suppose \( f \) and \( g \) are differentiable on \([a,b]\) and \( f' \) and \( g' \) are integrable on \([a,b]\). Prove that \( f'g \) and \( g'f \) are integrable on \([a,b]\), and prove that the integration by parts formula holds,

   \[ \int_a^b f'g \, dx = f(b)g(b) - f(a)g(a) - \int_a^b g'f \, dx. \]

2. Suppose \( \{a_n\}_{n=1}^\infty \) is a sequence of members of \([a,b]\) converging to \( x_0 \) in \([a,b]\). If \( f \) is bounded on \([a,b]\) and continuous on \([a,b]\) except at \( x_0 \) and the points of the sequence \( \{a_n\}_{n=1}^\infty \), prove that \( f \) is integrable on \([a,b]\).

3. Suppose \( f : R \to R \) is continuous and has period \( p \), so that \( f(x + p) = f(x) \) for all \( x \in R \). Show that \( \int_x^{x+p} f(t) \, dt \) is independent of \( x \) in that, for all \( x, y \),

   \[ \int_x^{x+p} f(t) \, dt = \int_y^{y+p} f(t) \, dt. \]

4. Suppose that \( f : [a, b] \to R \) is continuous, and define a function \( F(t) = \int_t^b f \, dx \) for all \( t \in [a, b] \). Show that \( F \) is differentiable on \([a, b]\) and find \( F'(t) \).