1. Determine whether each series converges and justify briefly.

(a) \( \sum_{n=1}^{\infty} \frac{1}{(n+1)(2n-1)} \)

Solution: \( \frac{1}{(n+1)(2n-1)} < \frac{1}{n^2} \leq \frac{1}{n^2} \); the last step holds because \( 1 \leq n, \ -1 \geq -n, \ 2n-1 \geq 2n-n = n \), \( \frac{1}{2n-1} \leq \frac{1}{n} \); \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a convergent \( p \)-series, so the series converges by the Comparison Test.

(b) \( \sum_{n=1}^{\infty} \frac{n}{n+1} \)

Solution: Since the sequence \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \) converges to 1, this series diverges by Cor. 6.2.

(c) \( \sum_{k=1}^{\infty} \frac{2}{3k} \)

Solution: Suppose this series converges. Then by Theorem 6.3, the series \( \sum_{n=1}^{\infty} \left( \frac{3}{2} \right) \frac{2}{3k} = \sum_{n=1}^{\infty} \frac{1}{k} \) converges, which is a contradiction because the harmonic series diverges. Therefore this series diverges.

(d) \( \sum_{m=1}^{\infty} \frac{\sqrt{m+1}-\sqrt{m}}{m} \)

Solution: \( \frac{\sqrt{m+1}-\sqrt{m}}{m} = \frac{1}{m(\sqrt{m+1}+\sqrt{m})} < \frac{1}{m(2\sqrt{m})} = \frac{1}{2m^{3/2}} \), and \( \sum_{m=1}^{\infty} \frac{1}{2m^{3/2}} = (1/2) \sum_{m=1}^{\infty} \frac{1}{m^{3/2}} \) is a convergent \( p \)-series. By the Comparison Test, \( \sum_{m=1}^{\infty} \frac{1}{\sqrt{m+1}+\sqrt{m}} \) converges.

(e) \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}} \)

Solution: \( \frac{1}{\sqrt{k+1}+\sqrt{k}} \geq \frac{1}{2\sqrt{k+1}} \geq \frac{1}{2(k+1)} \) \( \geq \frac{1}{2(2k)} = \frac{1}{4k} \), and \( \sum_{n=1}^{\infty} \frac{1}{4k} = (1/4) \sum_{n=1}^{\infty} \frac{1}{k} \) is a divergent \( p \)-series. By the Comparison Test, \( \sum_{k=1}^{\infty} \frac{1}{\sqrt{k+1}+\sqrt{k}} \) diverges.

(f) \( \sum_{k=1}^{\infty} \frac{1}{1+a^k} \) where \(-1 < a < 1\)

Solution: Since \( |a| < 1 \), the sequence \( \left\{ a^k \right\}_{k=1}^{\infty} \) converges to 0. Therefore the sequence \( \left\{ \frac{1}{1+a^k} \right\}_{k=1}^{\infty} \) converges to 1. By Corollary 6.2, the series \( \sum_{k=1}^{\infty} \frac{1}{1+a^k} \) must diverge.

2. Explain why the series converges, then find the sum.

\[ \sum_{n=2}^{\infty} \left( \frac{5}{2^n} - \frac{3}{5^n} \right) \]
Solution: Using Proof #3, since
\[ \sum_{n=2}^{\infty} \left( \frac{5}{2^n} - \frac{3}{5^n} \right) = 5 \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n - 3 \sum_{n=2}^{\infty} \left( \frac{1}{5} \right)^n, \]
and \( \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n \) and \( \sum_{n=2}^{\infty} \left( \frac{1}{5} \right)^n \) are convergent geometric series, the series converges. We have
\[ \sum_{n=2}^{\infty} \left( \frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n - (1/2)^0 - (1/2)^1 = \frac{1}{1-1/2} - 1 - 1/2 = 1/2 \]
and
\[ \sum_{n=2}^{\infty} \left( \frac{1}{5} \right)^n = \sum_{n=0}^{\infty} \left( \frac{1}{5} \right)^n - (1/5)^0 - (1/5)^1 = \frac{1}{1-1/5} - 1 - 1/5 = 1/20, \]
so \( \sum_{n=2}^{\infty} \left( \frac{5}{2^n} - \frac{3}{5^n} \right) = 5(1/2) - 3(1/20) = 47/20. \)

3. Explain why the series diverges.
\[ \sum_{n=1}^{\infty} \left( \frac{2}{n} + \frac{1}{2^n} \right) \]

Solution: We have \( \frac{2}{n} + \frac{1}{2^n} > \frac{2}{n} \) and \( \sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n} \) is a divergent \( p \)-series. All terms are positive, so we may apply the Comparison Test to conclude that \( \sum_{n=1}^{\infty} \left( \frac{2}{n} + \frac{1}{2^n} \right) \) diverges.
Proofs

For the remaining problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Prove that \( \sum_{n=1}^{\infty} (a_n - a_{n+1}) \) converges iff the sequence \( \{a_n\}_{n=1}^{\infty} \) converges.

**Solution:** Let \( S_n = \sum_{i=1}^{n} (a_i - a_{i+1}) \), so the series \( \sum_{n=1}^{\infty} (a_n - a_{n+1}) \) converges if and only if the sequence \( \{S_n\}_{n=1}^{\infty} \) converges. But

\[
S_n = (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_n - a_{n+1}) = a_1 - a_{n+1},
\]

so \( \{S_n\}_{n=1}^{\infty} = \{a_1 - a_{n+1}\}_{n=1}^{\infty} = \{a_1\}_{n=1}^{\infty} - \{a_{n+1}\}_{n=1}^{\infty} \). Clearly the sequence \( \{a_1\}_{n=1}^{\infty} \) converges, so \( \{S_n\}_{n=1}^{\infty} \) converges if and only if \( \{a_n\}_{n=1}^{\infty} \) converges.

2. Prove the Limit Comparison Test: Suppose \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series of positive terms such that \( \left\{ \frac{a_n}{b_n} \right\}_{n=1}^{\infty} \) converges to \( L \neq 0 \). Then \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) either both diverge or both converge.

**Solution:** (i) Suppose the series \( \sum_{n=1}^{\infty} b_n \) converges. Since the sequence \( \left\{ \frac{a_n}{b_n} \right\}_{n=1}^{\infty} \) converges, it is bounded; write \( \frac{a_n}{b_n} \leq M \). Then \( a_n \leq M b_n \). Applying the Comparison Test, we see that \( \sum_{n=1}^{\infty} a_n \) must converge.

(ii) Suppose the series \( \sum_{n=1}^{\infty} b_n \) diverges. Since \( L \neq 0 \), there exists \( m \) with \( 0 < m < L \) and \( \frac{a_n}{b_n} \geq m \) for all \( n \). Then \( a_n \geq m b_n \), so by the Comparison Test \( \sum_{n=1}^{\infty} a_n \) diverges.

3. Prove Theorem 6.3. That is, suppose \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge, and let \( \alpha, \beta \in \mathbb{R} \). Then \( \sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) \) converges, and

\[
\sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n.
\]

**Solution:** Let \( S_n = \sum_{i=1}^{n} a_n \) and \( T_n = \sum_{i=1}^{n} b_n \) be the partial sums for the series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \). Since \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) converge, the sequences \( \{S_n\}_{n=1}^{\infty} \) and \( \{T_n\}_{n=1}^{\infty} \) converge, say to \( A \) and \( B \) respectively. By Theorem 1.9, the sequence \( \{\alpha S_n\}_{n=1}^{\infty} \) converges to \( \alpha A \) and the sequence \( \{\beta T_n\}_{n=1}^{\infty} \) converges to \( \beta B \). By Theorem 1.8, the sequence \( \{\alpha S_n + \beta T_n\}_{n=1}^{\infty} \) converges to \( \alpha A + \beta B \). But this is the sequence of partial sums for the series \( \sum_{n=1}^{\infty} (\alpha a_n + \beta b_n) \), so this series must converge to \( \alpha A + \beta B = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n \).

4. (a) Prove that if \( \sum_{n=1}^{\infty} a_n \) converges absolutely, then \( \sum_{n=1}^{\infty} a_n^2 \) converges.

**Solution:** Since the series \( \sum_{n=1}^{\infty} |a_n| \) converges, the sequence \( \{|a_n|\}_{n=1}^{\infty} \) must converge to 0 by Corollary 6.2. In particular, there exists some \( N \) such that whenever \( n \geq N \), \( |a_n| < 1 \). Then for \( n \geq N \), \( a_n^2 \leq |a_n| \). Denote the partial sums of \( \sum_{n=1}^{\infty} |a_n| \) by

\[
S_n = \sum_{i=1}^{n} |a_n|.
\]
and denote the partial sums of $\sum_{n=1}^{\infty} a_n^2$ by

$$T_n = \sum_{i=1}^{n} a_i^2.$$  

Since $\sum_{n=1}^{\infty} |a_n|$ converges, the sequence $\{S_n\}_{n=1}^{\infty}$ converges, so it is bounded above, say by $M$. Note that the sequence $\{T_n\}_{n=1}^{\infty}$ is an increasing sequence, since each $a_n^2 > 0$. We will show that $\{T_n\}_{n=1}^{\infty}$ is bounded above by $M + a_1^2 + a_2^2 + \ldots + a_N^2$. When $n < N$,

$$T_n = a_1^2 + \ldots + a_n^2 \leq a_1^2 + \ldots + a_N^2 \leq a_1^2 + \ldots + a_N^2 + M.$$  

Consider $T_n$ for $n \geq N$.

$$T_n = a_1^2 + \ldots + a_N^2 + a_{N+1}^2 + \ldots + a_n^2 \leq a_1^2 + \ldots + a_N^2 + |a_{N+1}| + \ldots + |a_n| \leq a_1^2 + \ldots + a_N^2 + S_n \leq a_1^2 + \ldots + a_N^2 + M.$$  

Since the sequence $\{T_n\}_{n=1}^{\infty}$ is increasing and bounded above, by the Monotone Convergence Theorem (Theorem 1.16) the sequence is convergent. Since its sequence of partial sums is convergent, the sequence $\sum_{n=1}^{\infty} a_n^2$ is convergent.

(b) Determine whether the following statement is true or false, then prove it.

If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Solution: This statement is false. A counterexample is $a_n = (-1)^n/n$. Then

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent $p$-series, but $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series and thus divergent.

(c) Determine whether the following statement is true or false, then prove it.

If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Solution: This statement is false. A counterexample is $a_n = (-1)^n/\sqrt{n}$. The series $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test, but the series $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which is divergent.