Due date: Friday, Aug. 2

**Short Answer**
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. Give an example of a series for which Theorem 6.8 yields a conclusion but Theorem 6.9 does not.

2. Determine whether the following series converge. Minimal justification (e.g. name the theorem, briefly explain why it applies) is required.
   
   (a) \( \sum_{n=1}^{\infty} n^p p^n, \quad p > 0 \)
   
   (b) \( \sum_{n=1}^{\infty} (\sqrt{n} - 1)^n \)

3. Determine whether each series converges absolutely, converges conditionally, or diverges. Minimal justification (e.g. name the theorem, briefly explain why it applies) is required.
   
   (a) \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2 - 5n + 1} \)
   
   (b) \( \sum_{n=1}^{\infty} \frac{(-1)^n (n-5)}{n^3 - 7n - 9} \)

4. Use Theorem 6.9 to determine the values of \( r \) for which \( \sum_{n=0}^{\infty} nr^n \) converges.
Proofs

For the remaining problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Prove the following version of the Root Test:
   If \( \sum_{n=1}^{\infty} a_n \) is an infinite series such that \( \{ \sqrt[n]{|a_n|} \}_{n=1}^{\infty} \) converges to \( L \), then
   (i) if \( L < 1 \), the series converges absolutely; and
   (ii) if \( L > 1 \), the series diverges.

2. Let \( \sum_{k=1}^{\infty} a_k \) be a conditionally convergent series.
   (a) Show that there is a rearrangement of this series so that given any \( M \in \mathbb{R} \), there exists \( N \in J \) such that whenever \( n \geq N \), we have
   \[
   \sum_{k=1}^{n} a_k \geq M.
   \]
   (b) Show that there is a rearrangement of this series for which the sequence of partial sums \( \{ S_n \}_{n=1}^{\infty} = \{ \sum_{k=1}^{n} a_k \}_{n=1}^{\infty} \) is bounded but does not converge.

3. (a) Let \( \sum_{n=1}^{\infty} a_n \) be an infinite series and \( \{ n_k \}_{k=1}^{\infty} \) a subsequence of the sequence of positive integers. Prove that if \( \sum_{n=1}^{\infty} a_n \) converges absolutely, then \( \sum_{k=1}^{\infty} a_{n_k} \) converges absolutely.
   (b) Suppose that \( \sum_{n=1}^{\infty} a_n \) converges conditionally. What may we conclude about the convergence of \( \sum_{n=1}^{\infty} a_{n_k} \)? Prove it.

4. Consider the following rearrangement of the series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \):
   \[
   1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots.
   \]
   Let \( \{ S_n \} \) be the sequence of partial sums for \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \) and let \( \{ T_n \}_{n=1}^{\infty} \) be the sequence of partial sums for the rearrangement. Show that \( T_{3n} = S_{2n}/2 \) for each \( n \). What does this imply about the sums of the two series? Prove it.