Short Answer
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. For each function, determine what value \( f(x_0) \) should be assigned in order to make the function continuous at \( x_0 \), if it is possible to do so.
   (a) \( f : [0, 1] \to \mathbb{R}, f(x) = x - 1 \) for \( 0 < x \leq 1 \); \( x_0 = 0 \)
   (b) \( f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \) for \( x < 0 \) and \( f(x) = x + 1 \) for \( x > 0 \); \( x_0 = 0 \)
   (c) \( f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x^2 - 9}{x + 3}, x_0 = -3 \)

2. Rewrite each statement using either the universal quantifier “for any,” or the existential quantifier “there exists.”
   (a) If \( n \) and \( m \) are even integers, then \( m + n \) is an even integer.
   (b) A real number can be greater than 2.
   (c) A multiple of 6 is always a multiple of 3.
   (d) A real-valued function which is continuous at 0 is not necessarily differentiable at 0.
   (e) A triangle may be equilateral.

3. Write the negation of each statement from the previous problem. Use either the universal quantifier “for any,” or the existential quantifier “there exists.” (Since all the statements are true as stated, all of their negations will be false. Use this to check your work!)

4. The following statements contain nested quantifiers, that is, one quantifier followed by another. Rewrite each statement using the universal quantifier “for any,” and the existential quantifier “there exists.” There should be at least two quantifiers in each statement.
   (a) Any neighborhood \( P \) of \( x \) contains an interval of the form \((x - \epsilon, x + \epsilon)\).
   (b) For some \( \delta > 0 \), if \( |x - x_0| < \delta \) then \( |f(x)| < \epsilon \).
   (c) Given \( \epsilon > 0 \), we may find \( N \in J \) such that when \( n \geq N \), we have \( |a_n - 1| < \epsilon \).

5. Write the negation of each statement from the previous problem. Use the universal quantifier “for any” and the existential quantifier “there exists.”
Proofs
For the remaining problems, a proof is required. In particular, a complete solution
must be stated in sentence form with appropriate justification for each step.

1. State and prove a lemma similar to Lemma 2.7 for decreasing functions.

2. Let \( f : [a, b] \to \mathbb{R} \) be monotone. Prove that \( f \) has a limit at both \( a \) and \( b \) in two
   steps.
   (a) Prove that if \( f \) is decreasing, then \( f \) has a limit at \( a \) and \( b \) iff \( -f \) has a limit at
   \( a \) and \( b \).
   (b) Prove that if \( f \) is increasing, then \( f \) has a limit at both \( a \) and \( b \).

3. Using the definition of a continuous function, prove that \( f(x) = x^2 + x - 1 \) is
   continuous at \( x = 2 \).

4. Using the definition of a continuous function, prove that the function \( f \) is not
   continuous at \( x_0 = 0 \), where \( f \) is defined by \( f(x) = x + 1 \) for \( x \leq 0 \) and \( f(x) = 2x \)
   for \( x > 0 \).

5. Using the definition of a continuous function, prove that \( f(x) = x^3 + 1 \) is
   continuous.

6. Suppose that \( f : E \to \mathbb{R} \) is continuous at \( x_0 \), and \( x_0 \in F \subset E \). Define \( g : F \to \mathbb{R} \)
   by \( g(x) = f(x) \) for all \( x \in F \).
   (a) Prove that \( g \) is continuous at \( x_0 \).
   (b) Show by example that the continuity of \( g \) at \( x_0 \) need not imply the continuity
       of \( f \) at \( x_0 \).