Due date: Wednesday, July 10

Short Answer
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. Restate the definition of uniform continuity using the words “for any” each time a universal quantifier appears, and using the words “there exists” each time an existential quantifier appears.
   **Solution:** Let \( f : D \to \mathbb{R} \). Then \( f \) is uniformly continuous iff for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( x, y \in D \) such that \( |x - y| < \delta \), we have \( |f(x) - f(y)| < \epsilon \).

2. State the negation of the definition of uniform continuity using the words “for any” each time a universal quantifier appears, and using the words “there exists” each time an existential quantifier appears.
   **Solution:** Let \( f : D \to \mathbb{R} \). Then \( f \) is not uniformly continuous iff there exists \( \epsilon > 0 \) such that, given any \( \delta > 0 \), there exist \( x, y \in D \) with \( |x - y| < \delta \) and \( |f(x) - f(y)| \geq \epsilon \).

3. Consider the function \( f : (0, 1) \to \mathbb{R} \) defined by \( f(x) = 1/x \). Use Theorem 3.5 to show that \( f \) is not uniformly continuous. That is, find an accumulation point of the domain of \( f \) at which \( f \) does not have a limit.
   **Solution:** The point 0 is an accumulation point of the domain \((0, 1)\). Suppose that \( f \) has a limit at 0. Then, since the sequence \( \{1/n\}^\infty_{n=1} \) converges to 0 and has points lying in the domain \((0, 1)\), the sequence \( \{f(1/n)\}^\infty_{n=1} = \{n\}^\infty_{n=1} \) must converge. This cannot be; for one thing, this sequence in unbounded. This gives a contradiction, so \( f \) must not have a limit at 0.

Proofs
For the following problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Suppose \( f, g : D \to \mathbb{R} \) are both continuous on \( D \). Define \( h : D \to \mathbb{R} \) by \( h(x) = \max\{f(x), g(x)\} \). Show that \( h \) is continuous on \( D \).
   **Solution:** Let \( \epsilon > 0 \) be given. We will show that \( h \) is continuous at a point \( a \in D \). Since \( f \) is continuous at \( a \), there exists \( \delta_f > 0 \) such that whenever \( x \in D \) and \( |x - a| < \delta_f \), we have \( |f(x) - f(a)| < \epsilon \).
   Since \( g \) is also continuous at \( a \), there exists \( \delta_g > 0 \) such that whenever \( x \in D \) and \( |x - a| < \delta_g \), we have \( |g(x) - g(a)| < \epsilon \).
   Select \( \delta = \min\{\delta_f, \delta_g\} \). Let \( x \in D \) with \( |x - a| < \delta \).
Without loss of generality, suppose \( h(a) = f(a); \) that is, suppose \( f(a) \geq g(a). \) We know that \( h(x) \) will be either \( f(x) \) or \( g(x). \) If \( h(x) = f(x), \) then

\[
|h(x) - h(a)| = |f(x) - f(a)| < \epsilon,
\]

so \( h \) is continuous at \( a \) in this case. Instead, suppose \( h(x) = g(x), \) which will occur if and only if \( g(x) \geq f(x). \) Then we have

\[
-\epsilon < f(x) - f(a) \leq g(x) - f(a) \leq g(x) - g(a) < \epsilon,
\]

so \( |g(x) - f(a)| < \epsilon. \) Since \( |h(x) - h(a)| = |g(x) - f(a)|, \) this shows that \( h \) is continuous at \( a. \)

Since we have shown that \( h \) is continuous at all \( a \in D, \) \( h \) is a continuous function.

2. Suppose \( f : D \to R \) is continuous at \( x_0 \in D. \) Prove that \( |f| : D \to R \) such that \( |f|(x) = |f(x)| \) is continuous at \( x_0. \)

**Solution:** Let \( \epsilon > 0 \) be given. Since \( f \) is continuous at \( x_0, \) there exists \( \delta \) such that if \( x \in D \) and \( |x - x_0| < \delta, \) then \( |f(x) - f(x_0)| < \epsilon. \) Then we have

\[
|f(x)| - |f(x_0)| \leq |f(x) - f(x_0)| < \epsilon,
\]

so the same value of \( \delta \) shows that \( f \) is continuous at \( x_0 \) as well.

3. Suppose \( f : D \to R \) with \( f(x) \geq 0 \) for all \( x \in D. \) Show that, if \( f \) is continuous at \( x_0, \) then \( \sqrt{f} \) is continuous at \( x_0. \)

**Solution:** Let \( \epsilon > 0 \) be given. If \( f(x_0) \neq 0, \) then set \( \epsilon' = \epsilon \sqrt{f(x_0)}. \) Since \( f \) is continuous at \( x_0, \) there exists \( \delta \) such that when \( x \in D \) and \( |x - x_0| < \delta, \) we have \( |f(x) - f(x_0)| < \epsilon \sqrt{f(x_0)}. \) Then we have

\[
|\sqrt{f(x)} - \sqrt{f(x_0)}| = \left| \frac{f(x) - f(x_0)}{\sqrt{f(x)} + \sqrt{f(x_0)}} \right| < \frac{|f(x) - f(x_0)|}{\sqrt{f(x_0)}} < \frac{\epsilon \sqrt{f(x_0)}}{\sqrt{f(x_0)}} = \epsilon.
\]

This shows that \( \sqrt{f} \) is continuous at \( x_0 \) if \( f(x_0) \neq 0. \)

If \( f(x_0) = 0, \) then set \( \epsilon' = \epsilon^2. \) Since \( f \) is continuous at \( x_0, \) there exists \( \delta \) such that when \( x \in D \) and \( |x - x_0| < \delta, \) then \( |f(x) - f(x_0)| < \epsilon^2; \) that is, \( |f(x)| < \epsilon^2. \) Then

\[
|\sqrt{f(x)} - \sqrt{f(x_0)}| = |\sqrt{f(x)}| < \epsilon^2 = \epsilon,
\]

so \( \sqrt{f} \) is continuous in this case also.

4. Show that \( f : (1, 5) \to R \) defined by \( f(x) = 2x^2 - x + 3 \) is uniformly continuous.

(Note that Theorem 3.8 does not apply.)

**Solution:** Let \( \epsilon > 0 \) be given. Select \( \delta = \epsilon/9. \) Suppose \( x, y \in (1, 5) \) with \( |x - y| < \epsilon/9. \) Note that since \( 1 < x < 5 \) and \( 1 < y < 5, \) we have \( 4 < 2x + 2y < 10 \) and \( 3 < 2x + 2y - 1 < 9. \) Therefore, we have

\[
|f(x) - f(y)| = |2x^2 - x - 2y^2 + y| = |(2x + 2y - 1)(x - y)| < 9\delta = \epsilon.
\]

It follows that \( f \) is uniformly continuous on \((1, 5).\)
5. Show that \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 2x^2 - x + 3 \) is not uniformly continuous by using the definition of uniform continuity.

**Solution:** In order to complete this proof, I will make several choices, so I will discuss the thinking that goes into the proof in hopes that this will make the choices clearer.

When planning the proof for this problem, I first observe that the parabola \( f(x) \) gets arbitrarily "steep," so it makes sense that the \( \delta \)-value attached to any value of \( \epsilon \), say \( \epsilon = 1 \) for example, will not be sufficient to keep the distance from \( f(x) \) to \( f(y) \) smaller than \( \epsilon \); to find \( x \)- and \( y \)-values that are within \( \delta \) of each other but whose images are not within a distance of 1, we will simply have to go far enough out from the vertex of the parabola.

In my proof, I will fix \( \epsilon = 1 \) (though any other fixed value would work just as well), and I will take \( \delta \) to be given. I will find \( x, y \) such that \( |x - y| < \delta \) and \( |f(x) - f(y)| \geq 1 \). For concreteness, I will choose to make \( x - y = \delta/2 \) (though any value smaller than \( \delta \) would work). I want to show that \( |f(x) - f(y)| \geq 1 \); that is, \( |(2x + 2y - 1)(x - y)| \geq 1 \). Since I will choose \( x - y = \delta/2 \), I can rewrite this all in terms of \( y \) only; this gives

\[
|f(x) - f(y)| = |(2(\delta/2 + y) + 2y - 1)(\delta/2)| \geq 1
\]

\[
|(\delta + 4y - 1)(\delta/2)| \geq 1
\]

\[
|\delta + 4y - 1| > 2/\delta
\]

As long as I choose \( y \) so that \( y \geq 1/4 \), we have

\[
\delta + 4y - 1 \geq 2/\delta
\]

\[
y \geq \frac{1}{2\delta} - \frac{\delta}{4} + \frac{1}{4}
\]

We can choose any value of \( y \) which satisfies both this inequality and also \( y \geq 1/4 \). I’ll choose \( y = \frac{1}{2\delta} + \frac{1}{4} \). Then since I decided that \( x - y = \delta/2 \), I’ll choose \( x = \frac{1}{2\delta} + \frac{1}{4} + \frac{\delta}{2} \).

Now I am ready to write the actual proof.

**Proof:** Suppose that \( f \) is uniformly continuous on \( \mathbb{R} \). Select \( \epsilon = 1 \). Then there exists a \( \delta > 0 \) such that whenever \( x, y \in \mathbb{R} \) and \( |x - y| < \delta \), we have \( |f(x) - f(y)| < 1 \). Consider \( x = \frac{1}{2\delta} + \frac{1}{4} + \frac{\delta}{2} \) and \( y = \frac{1}{2\delta} + \frac{1}{4} \). Note that \( |x - y| = \delta/2 < \delta \). Observe that

\[
|f(x) - f(y)| = |(2x^2 - x + 3) - (2y^2 - y + 3)| = |(2x + 2y - 1)(x - y)|
\]

\[
= \left| 2\left(\frac{1}{2\delta} + \frac{1}{4} + \frac{\delta}{2}\right) + 2\left(\frac{1}{2\delta} + \frac{1}{4}\right) - 1 \right| \left(\frac{\delta}{2}\right)
\]

\[
= \left| \frac{2}{\delta} + \frac{\delta}{2} \right| = \left| 1 + \frac{\delta^2}{2} \right| > 1.
\]

This gives a contradiction, so the function \( f \) is not uniformly continuous on \( \mathbb{R} \).
6. A function $f : R \to R$ is periodic iff there is a real number $h \neq 0$ such that $f(x + h) = f(x)$ for all $x \in R$. Prove that if $f : R \to R$ is periodic and continuous, then $f$ is uniformly continuous.

**Solution:** Consider a function $g : [0, 2h] \to R$ defined by $g(x) = f(x)$ for every $x \in [0, 2h]$. By Homework 5 Proof #6, we know that because $f$ is continuous at every point in $[0, 2h]$, $g$ must be as well. By Theorem 3.8, since $g$ is a continuous function on a compact domain, $g$ must be uniformly continuous.

Let $\epsilon > 0$ be given. Since $g$ is uniformly continuous, there exists $\delta_g$ such that whenever $x, y \in [0, 2h]$ with $|x - y| < \delta_g$, we have $|g(x) - g(y)| < \epsilon$.

Fix $\delta = \min\{\delta_g, h\}$. Let $x, y \in R$ with $|x - y| < \delta$, and suppose without loss of generality that $x \leq y$. Write $x = kh + x'$ where $k \in Z$ and $0 \leq x' < h$. Note that $f(x) = f(x')$ because $f$ is periodic with period $h$, and $f(x') = g(x')$ because $x' \in [0, 2h] = \text{dom } g$. Now write $y' = y - kh$. Observe that since $|x - y| < \delta \leq h$, and $x \leq y$, we have

$$0 \leq y - x \leq h$$
$$0 \leq (kh + y') - (kh + x') \leq h$$
$$0 \leq y' - x' \leq h$$
$$x' \leq y' \leq h + x'$$

Since $0 \leq x' < h$, this gives $0 \leq x' \leq y' \leq h + x' \leq 2h$, which shows that $y' \in [0, 2h]$. Note that since $|x - y| < \delta$, we have

$$|x' - y'| = |(x - kh) - (y - kh)| = |x - y| < \delta \leq \delta_g.$$ 

Now we have

$$|f(x) - f(y)| = |f(x') - f(y')| = |g(x') - g(y')| < \epsilon,$$

so $f$ is uniformly continuous.