Math 4603 Advanced Calculus I
Homework 7: Sections 3.3, 3.4

Due date: Friday, July 12

Short Answer
State the answer to each of the following questions. It is not necessary to write down any justification (though of course one would want to be able to explain how each solution was obtained).

1. Determine if each set $S$ is open. If it is open, give the value of $\epsilon$ such that a neighborhood of the form $(x - \epsilon, x + \epsilon)$ is contained in the set $S$ for a general element $x \in S$. If it is not open, give an element of the set for which such a neighborhood cannot be found.
   (a) $S = (0, 1)$
   (b) $S = [0, 1]$  
   (c) $S = (0, 1]$ 
   (d) $S = \mathbb{Z}$
   (e) $S = \mathbb{R}$
   (f) $S = \emptyset$
   (g) $S = \{1/n : n \in J\}$

2. Determine if each set from the previous problem is closed. If it is closed, list its accumulation points. If it is not closed, give an accumulation point which is not in the set $S$.

3. Show that each set is not compact by finding an open cover of each set which has no finite subcover.
   (a) $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$
   (b) $(0, 1)$

4. Find an interval of length 1 that contains a root of each equation.
   (a) $xe^x = 1$
   (b) $x^3 - 6x^2 + 2.826 = 0$
Proofs
For the remaining problems, a proof is required. In particular, a complete solution must be stated in sentence form with appropriate justification for each step.

1. Let $E \subset \mathbb{R}$. Prove that $E$ is closed if, for every $x_0$ such that there is a sequence $(x_n)_{n=1}^{\infty}$ of points of $E$ converging to $x_0$, it is true that $x_0 \in E$. In other words, prove that $E$ is closed if it contains all limits of sequences of members of $E$.

2. (a) Prove that every set of the form $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is open.
   (b) Prove that every set of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is closed.

3. Let $D \subset \mathbb{R}$, and let $D'$ be the set of accumulation points of $D$. Prove that $\overline{D} = D \cup D'$ is closed and that if $F$ is an closed set that contains $D$, then $\overline{D} \subset F$. ($\overline{D}$ is called the closure of $D$.)

4. If $D \subset \mathbb{R}$, then $x \in D$ is said to be an interior point of $D$ iff there is a neighborhood $Q$ of $x$ such that $Q \subset D$. Define $D^\circ$ to be the set of interior points of $D$. Prove that $D^\circ$ is open, and that if $S$ is any open set contained in $D$, then $S \subset D^\circ$. ($D^\circ$ is called the interior of $D$.)

5. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f(b) \leq y \leq f(a)$. Prove that there is $c \in [a, b]$ such that $f(c) = y$. (Note that this is a slightly different statement than the Intermediate Value Theorem. You may reference the IVT in your solution.)

6. Suppose that $f : [a, b] \to [a, b]$ is continuous. Prove that there is at least one fixed point in $[a, b]$; that is, there exists $x \in [a, b]$ such that $f(x) = x$. 